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THE UNIVERSITY OF ALBERTA

ELECTROMAGNETIC ANGULAR MOMENTUM

FLOW IN INDUCTION MACHINES

BY

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A THESIS

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ABSTRACT

The law of conservation of momentum is interpreted for an electromagnetic system. Using Maxwell's stress tensor and the concept of electromagnetic angular momentum flow an expression is obtained to calculate the instantaneous electromagnetic torque on a body in free space with a view to apply it to the theory of electrical machines.

The electromagnetic field problem of the double cylindrical machines is reduced to finding the transient solution of a vector wave equation. The method of the Laplace transform is used to develop a general solution of the equation. The solution is exact and may include any number of space harmonics and time functions associated with the driving frequencies and the characteristic roots of the electromagnetic wave system.

The machine volt-ampere equations are obtained by integrating the electric field intensities and the torque equation is deduced from the momentum concept. Quasi-static field quantities and the resulting machine equations of motion are also derived from the low argument approximations of the Bessel functions involved in the exact equations.

An exact wave theory analysis of an induction machine is developed in detail to study the production of torque in terms of a transfer of electromagnetic angular momentum across the air gap. The simpler case of the machine under steady state conditions is treated first and subsequently a solution is given for a particular type of

transient. It is found that the quasi-static approximation provides completely satisfactory quantitative information in the steady state, but the exact treatment is necessary to give a proper account of the torque at each instant during the course of the transient.

ACKNOWLEDGEMENT

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

NOMENCLATURE

(A) Subscripts and Superscripts

a .... Phase a quantity

b .... Phase b quantity

n .... Order of space harmonics and of Bessel functions

t .... Total

s .... Stator quantity

r .... Rotor quantity

ss .... Stator quantity observed from the stator co-ordinate
system

sr .... Rotor quantity observed from the stator co-ordinate
system

rr .... Rotor quantity observed from the rotor co-ordinate
system

rs .... Stator quantity observed from the rotor co-ordinate
system

(B) Analytic Quantities

a .... Rotor radius, m

$\vec{A}$ .... Vector potential in the time domain

A .... z-component of the vector potential in the time domain

$\bar{A}$ .... Vector potential in the frequency domain

b .... Stator radius, m

$\vec{B}$ .... Magnetic flux density, weber/m^2

c .... Velocity of light in free space, m/sec

$\vec{D}$ .... Electric displacement, c/m^2

$\vec{E}$ .... Electric field intensity, v/m

ε_o .... Permittivity of free space

x

f .... Frequency, c.p.s.

$\vec{f}$ .... Electromagnetic volume force density

g .... Air gap length, m

g_m .... Mechanical momentum density

g_{em} .... Electromagnetic linear momentum density

$g_{e\omega}$ .... Electromagnetic angular momentum density

G_ω .... Total electromagnetic angular momentum

h .... Planck's constant

$\vec{H}$ .... Magnetic field intensity, a/m

$\vec{i}_r, \vec{i}_\theta, \vec{i}_z$ .. Unit vectors of a system of cylindrical co-ordinates

i, $i(t)$ ... Instantaneous current, a

$i_1, i_1(t)$ .. Instantaneous linear current density, a/m

I^s .... Maximum amplitude of the stator current, a

I^r .... Maximum amplitude of the rotor current, a

I_1^s .... Maximum amplitude of linear current density in the stator, a/m

I_1^r .... Maximum amplitude of linear current density in the rotor, a/m

$\bar{I}(s)$ ...... Currents in frequency domain

$\bar{I}_1(s)$ ..... Linear current density in frequency domain

$I_n(z)$ ..... Modified Bessel Functions of the first kind and of order n

Im .... Imaginary part of

j .... $(-1)^{1/2}$

$\vec{J}_f$ .... Free current density, a/m^2

$\vec{J}^*$ .... Magnetic current density

$\vec{J}_t$ .... Total current density, a/m^2

J_m .... Moment of inertia of the rotating system

k .... s/c

$K_n(z)$ ..... Modified Bessel Functions of the second kind and of order n

l .... Machine axial length

L .... Inductance, henry

$\vec{n}$ .... Unit vector in the normal direction

m .... ω/c

m_r .... ω_r/c

m_s .... ω_s/c

M .... Maxwell's Stress Tensor

μ_o .... Permeability of free space

μ_r .... Relative permeability of iron

p .... Total differentiation with respect to time

P .... Power

P_{av}^r .... Average power with respect to the rotor co-ordinate system

P_{av}^S .... Average power with respect to the stator co-ordinate system

r .... Radial co-ordinate of a cylindrical system

R^S .... Stator resistance per phase

R^r .... Rotor resistance per phase

R_r .... Rotor surface resistivity

Re .... Real part of

s .... Complex number

$\vec{S}$ .... Poynting vector

S .... Surface

T .... Torque

T_{em}^{r} .... Average mechanical torque of electromagnetic origin on the rotor

$T_{em}^{r}(t)$ .... Instantaneous mechanical torque of electromagnetic origin on the rotor

T_{em}^{s} .... Average torque on the stator

$T_{em}^{s}(t)$ .... Instantaneous torque on the stator

v .... Voltage, v

v_E .... Energy velocity

v_ω .... Electromagnetic angular momentum velocity

w .... Energy density

W .... Total energy

$\bar{X}_n(s)$ ..... Functions of s substituted to replace expressions including modified Bessel Functions, independent of r

$\bar{X}_n'(r,s)$.... Functions of r and s substituted to replace expressions including modified Bessel Functions

$X_n'(\omega)$...... Functions of ω substituted to replace expressions including Bessel Functions

$X_n(r,\omega)$ ... Functions of r and ω substituted to replace expressions including Bessel Functions

z .... Axial co-ordinate of cylindrical system

Z .... Impedence

α.......... Real part of the complex frequency

β .... Imaginary part of the complex frequency

δ .... Conductivity

σ .... Resistivity

ϕ .... Rotor angular position with respect to the stator co-ordinate system

ψ .... Impedance angle

$\mathcal{J}$ .... Time constants

λ .... Wave length

ν .... Order of terms in the solution of wave equation

$\ddot{\omega}$ .... Angular frequency

ω_m .... Rotor angular velocity

ρ .... Charge density

θ .... Tangential co-ordinate in a cylindrical system

θ_p .... Phase angle

CHAPTER I

INTRODUCTION

1.1 Historical Development of the Concept of Electromagnetic Momentum

Newton's corpuscular theory of light led its adherents in the
early eighteenth century to associate momentum with radiant energy.
Many investigators[1] in that century directed powerful beams of light
on delicately suspended bodies and tried to detect a pressure due to
the impulse of the corpuscles. Motions were observed but the experi-
ments were inconclusive due to thermal effects which were not properly
understood at that time. From a theoretical study Euler[1,2] inferred
the existence of light pressure in 1748. He further accounted for
the tails of comets by supposing that solar rays, impinging on the
atmosphere of a comet drive off the more subtle of its particles.

In the late nineteenth century, J.C. Maxwell gave his celebrated
electromagnetic theory which regards light and radiant heat as
phenomena of electromagnetic wave motion. On the basis of his theory
he[3] deduced in 1873 that in a medium in which waves are propagated
there is a pressure in the direction normal to the waves and numer-
ically equal to the energy density. Thus, if a beam of radiation is
incident perpendicularly on a perfectly reflecting metallic sheet the
radiation pressure is equal to the density of energy in the region
immediately outside the metal. In 1876 Adolfo Bartoli[1] argued that
if radiant energy is transported from a cold body to a hot one by

means of a moving mirror, the second law of thermodynamics would be violated unless a pressure were exerted on the mirror by the light. In 1900, Planck's quantum theory[4] further reinforced the association of momentum with radiation.

The first reliable experimental verifications of Maxwell's theoretical result on radiation pressure were done by Lebedew[5] in 1899 and E.F. Nichols and G.F. Hull[6] in 1901. N. Carrara and P. Lombardini[7], in 1949, detected radiation pressure of centimeter waves by using a 725A-type magnetron and a rectangular wave guide system. In 1952, Cullen and his associates [8,9] developed an apparatus for the absolute measurement of microwave power in terms of radiation pressure. Their instrument consisted essentially of a small vane, suspended in rectangular wave guide carrying an H_{01} mode.

If linear momentum could be associated with a plane polarized wave, it was a small step to associate angular momentum with a circularly polarized wave. This was first done by Sadowsky[10] in 1899, and later by J.H. Poynting[11] in 1909 and Epstein[12] in 1914. With such an angular momentum they also predicted an associated torque on a material body. Dr. Poynting compared the angular momentum with that of a revolving shaft.

In 1935, R.A. Beth[13] passed circularly polarized light through a doubly refracting plate which changed the state of polarization and he was able to detect a torque, thus making the first experimental verification of the existence of electromagnetic angular momentum. N. Carrara[14] reported experimental works at microwave frequency in

1949. Recently, at the University of Alberta, F.S. Chute[15] measured within experimental error the reaction torque on an antenna that radiated a net angular momentum at radio frequency.

Historically, aether[16] played an important role in man's imagination in developing different philosophies, and its model[1] varied from age to age. The momentum concept associated with electromagnetic theory was first mathematically developed by Sir J.J. Thomson[17], a follower of Maxwell, in the last decade of the nineteenth century. He adopted the view that aether was the vehicle of momentum in apparently free space and in an electromagnetic field problem the aetheral momentum must be taken into account for Newton's third law of motion and the principle of conservation of momentum to be valid.

The hypothesis that aether is a store house of momentum was further developed by Poincare[17,18], who in 1900, referring to Thomson's expression that in free aether the electromagnetic momentum is $(1/c^2)$ times the Poynting flux of energy, suggested that electromagnetic energy might possess mass density equal to $(1/c^2)$ times the energy density.

The failure of the Michelson-Morley experiment led to the abandonment of the idea of a material aether and resulted in the ultimate development of the special theory of relativity by Einstein. An essential feature of this theory is the association of mass with energy and the enormous success of the theory is probably the strongest argument in favour of the concept of electromagnetic angular momentum.

1.2 Survey of the Field Theory of Electrical Machines

Machine engineers have attempted from time to time to adapt Maxwell's equations to study different aspects of electrical machines. Modern generalized machine analysis includes the formulation of machine differential equations in certain reference system in terms of the lumped parameters of circuit theory and the solution of these equations for specified load and terminal conditions. The quasi-static solution of Maxwell's equation gives the approximate electro-magnetic fields in different parts of the machine. The magnetic fields thus obtained are utilized to define inductance parameters through the concept of energy storages or flux linkages. The differential equations are then formed by using basic force laws or Euler-Lagrange equation of classical mechanics.

For specified conditions the machine equations may be treated by various topographical methods[19] including an equivalent circuit approach for easy cases or by the more generalized tensorial methods[20,21]. These methods may collectively be called the "system methods" and most of the time there is an interplay between them. The quasi-static field components are sometimes used in connection with Poyntings vector to calculate the power transfer through the air gap of the machine.

Historically, perhaps Creedy[22] was the first to discuss wave phenomena in the dynamo-electric machine in 1917. Hague[23] published a series of papers between 1917 and 1926, where he discussed electro-magnetic fields and forces in various regions of a machine.

Ollendorf[24] was the first to use vector potentials in machine theory.

The application of Poynting's vector to energy calculations in low frequency power apparatus and systems was first given by Slepian[25] in 1919. He tried to give a picture of the way the insulating medium supporting electromagnetic waves plays a vital role in leading the electrical power to where it is consumed. In 1942, Slepian[26] again, in a very ingenious derivation of the energy flow theorem gave nine alternative forms of the power flow vector. Alger and Oney[27,28] studied Poynting's energy flow in induction machines and Hawthorne[29] did the same in synchronous machines. Alger and Oney tried to visualize the machine as an array of magnetized air spaces in which intense and turbulent energy concentrations exist.

In 1954, Mishkin[30] obtained approximate solutions of Maxwell's equations for an idealized model of an induction motor. His model replaced the toothed stator and rotor by continuous, but inhomogeneous and anisotropic regions, having averaged properties. Cullen and Barton[31] analyzed the induction machine in 1958, by making use of the wave impedance concept and employing a transmission line analogue. The sleeve-rotor machine was analyzed by Guilford[32] in 1962. Besides solving Laplace's equation in the air gap, he solved the diffusion equation for the conducting sleeve.

Professor Saunders'[33,34] contribution to the field theory of electrical machines is quite significant. Using probable practical conductor distributions in a double cylindrical structure he generated surface harmonic current sheets and obtained quasi-static solutions

of Maxwell's equations for the air gap fields. In his field equations
there were provisions to include time harmonics in the surface curr-
ents. A Lagrangian formulation was then used to find the equations
of motion. Recently, Nasar[35] published a review paper on the electro-
magnetic theory of electrical machines, giving a brief digest of exist-
ing information in a unified form by classifying diverse methods of
analysis. A large bibliography on the subject is also provided.

1.3 Background of the Electromagnetic Angular Momentum Concept
     Applied to Induction Machine Theory

In 1958, Cullen and Barton[31] mentioned that the theory of the
squirrel cage induction machine is closely related to the phenomenon
of radiation pressure. They tried to draw an analogy between the
force acting on the rotor of an induction motor and the radiation
stress of a surface wave, obliquely incident on an absorbing surface.

In two recent papers, Professor Barlow[36,37] refers to the induc-
tion motor as a device in which torque is produced by the absorption
of electromagnetic angular momentum. The inference is that electro-
magnetic angular momentum is generated by the stator winding and is
propagated across the air gap to the rotor, in much the same way as
angular momentum is propagated in a waveguide supporting a circularly
polarized wave.

The above three references mention the mathematical relation
$P = T\omega$, where P is the power, T is the torque and ω is the angular
velocity of the field components. At this point it is appropriate to
note that this relation is very well known to induction machine engin-

eers; but they derived it without ever associating momentum properties with the electromagnetic field. The concept of electromagnetic angular momentum in relation to induction machine torque production has not yet been developed in the mathematical sense and it is a problem to be discussed in this thesis.

1.4 Conservation of Energy and Momentum in Electromagnetic Systems –
 Poynting's Equation and Maxwell's Stress Tensor

Two of the fundamental laws of classical physics are the conservation of energy and momentum. Since energy and momentum of material bodies can be affected by an interaction with an electromagnetic field, it is convenient and proper to associate an electromagnetic energy and an electromagnetic momentum with the field in such a way as to preserve the conservation laws.

In the case of energy, a few simple manipulations of Maxwell's equations lead to the famous Poynting's equation, namely

$$ -\int_S (\vec{E} \times \vec{H}) \cdot \vec{n} dS \;=\; \int_V \vec{J}_f \cdot \vec{E} dV \;+\; \int_V (\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}) dV \qquad ...1.1 $$

The second term on the right-hand side of equation (1.1) is interpreted as the rate of increase of electric and magnetic energy within the volume; the first term includes the power dissipated as Joule heat and any other electromagnetic energy conversion. From the conservation of energy the left-hand side of equation (1.1) then represent the inward flow of electromagnetic energy across the surface bounding the volume.

A similar profit and loss account can be made for momentum. Con-

sider the electromagnetic volume force density given by [38]

$$\vec{f} = \rho_t \vec{E} + \vec{J}_t \times \vec{B} \qquad \qquad \ldots 1.2$$

where ρ_t is the sum of free and polarization charges and $\vec{J}_t$ includes free, polarization and amperian current densities. The electromagnetic force density given by equation (1.2) may be equated to the time rate of change of the mechanical momentum density g_m, that is:

$$\frac{dg_m}{dt} = \rho_t \vec{E} + \vec{J}_t \times \vec{B} \qquad \qquad \ldots 1.3$$

Using Maxwell's equations it may be shown that

$$\rho_t \vec{E} + \vec{J}_t \times \vec{B} = \epsilon_0 \vec{E} \, \nabla.\vec{E} + \frac{\nabla \times \vec{B} \times \vec{B}}{\mu_0} + \epsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} \qquad \ldots 1.4$$

since
$$\epsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} = \epsilon_0 E \times \frac{\partial \vec{B}}{\partial t} - \frac{\partial}{\partial t}(\epsilon_0 \, \vec{E} \times \vec{B})$$

$$= -\epsilon_0 \vec{E} \times \nabla \times \vec{E} - \frac{\partial}{\partial t}(\epsilon_0 \vec{E} \times \vec{B})$$

and
$$\frac{\vec{B} \nabla. \vec{B}}{\mu_0} \equiv 0$$

one may write

$$\vec{f} + \frac{\partial}{\partial t}(\epsilon_0 \vec{E} \times \vec{B}) = \epsilon_0 \left[\vec{E} \nabla.\vec{E} - \vec{E} \times \nabla \times \vec{E} \right]$$

$$\left. + \frac{1}{\mu_0} \left[\vec{B} \, \nabla. \vec{B} - \vec{B} \times \nabla \times \vec{B} \right] \right\} \qquad \ldots 1.5$$

Introduce the symmetric tensor M of rank two, known as Maxwell's stress tensor, defined in dyadic form by

$$M = \epsilon_0 \vec{E}\vec{E} + \frac{1}{\mu_0} \vec{B}\vec{B} - \frac{1}{2}(\epsilon_0 \vec{E}.\vec{E} + \frac{1}{\mu_0}\vec{B}.\vec{B})U \qquad \ldots 1.6$$

where U is the fundamental tensor whose components are $\delta_{\alpha\beta}$, the Kronecker delta.

In appendix (1) it is shown that the right-hand side of equation (1.5) is equal to the divergence of M, where

$$\text{div } M = (M_{\alpha\beta}).\overleftarrow{\nabla} \triangleq i_\alpha \sum_{\beta=1}^{3} \frac{\partial M_{\alpha\beta}}{\partial\beta}, \quad \alpha = 1,2,3. \qquad \ldots 1.7$$

Hence it follows that

$$\text{div } M = \vec{f} + \frac{\partial}{\partial t}(\epsilon_o \vec{E} \times \vec{B}) \qquad \ldots 1.8$$

Taking the volume integral of both sides of (1.8) and using the divergence theorem of Gauss applied to tensor of rank two, one may write

$$\int_S M.\vec{n}\,dS = \int_V (\rho_t \vec{E} + \vec{J}_t \times \vec{B})\,dV + \frac{\partial}{\partial t}\int_V (\epsilon_o \vec{E} \times \vec{B})\,dV \qquad \ldots 1.9$$

Let the surface S now recede to infinity or at least far enough to yield a negligible contribution so that the left-hand side of equation (1.9) is zero. Combining this with equation (1.2) one obtains

$$\frac{d}{dt}\int_V (g_m + \epsilon_o \vec{E} \times \vec{B})\,dV = 0 \qquad \ldots 1.10$$

Therefore,

$$\int_V (g_m + \epsilon_o \vec{E} \times \vec{B})\,dV = \text{ Constant}$$

The quantity

$$g_{em} = \epsilon_o \vec{E} \times \vec{B} \qquad \ldots 1.11$$

is dimensionally a momentum per unit volume and may be referred to as the volume density of electromagnetic linear momentum. In an electromagnetic system, the sum of mechanical and electromagnetic momentum is conserved -- the mechanical momentum is not necessarily conserved.

There is a marked similarity between equations (1.1) and (1.9). The first integral on the R.H.S. of (1.9) is the total mechanical force which by Newton's second law is the rate at which mechanical momentum is produced. Since the integrand is expressed in terms of electrical quantities it may be interpreted as the rate at which electromagnetic momentum is converted to another form of momentum. The second integral on the R.H.S. gives the rate of increase of electromagnetic momentum within the volume. Consequently, by the law of conservation of momentum the L.H.S. gives the inward flux of electromagnetic momentum through the bounding surface, S.

If Maxwell's equations are written in symmetrical form [39] by defining a magnetic current density $\vec{J}_m^* = \frac{\partial}{\partial t}(\mu_0 \vec{M})$ and a magnetic charge density $\rho^* = -\text{div}\mu_0 M$, then the force

$$\vec{f} = \rho\vec{E} + \vec{J}\times\mu_0\vec{H} + \rho^*H - \vec{J}^* \times \epsilon_0\vec{E} \qquad ...1.2A$$

where
$$\rho = \rho_f - \nabla\cdot\vec{P}$$

Then
$$\vec{J} = \vec{J}_f + \frac{\partial\vec{P}}{\partial t}$$

$$M = \epsilon_0\vec{E}\vec{E} + \mu_0\vec{H}\vec{H} - \frac{1}{2}(\epsilon_0\vec{E}\cdot\vec{E} + \mu_0\vec{H}\cdot\vec{H})U \qquad ...1.6A$$

and
$$g_{em} = \mu_0\epsilon_0(\vec{E}\times\vec{H}) \qquad ...1.11A$$

The difference in the two definitions of momentum density given

by (1.11) and (1.11A) is due to the fact that the mechanical force
density acting inside a material medium has been given by different
expressions. There would appear to be no experimental procedure by
which this problem can be resolved. It is important to note that
in free space the two approaches are identical. In this thesis only
the overall forces on material bodies (rotor and stator) are con-
sidered, and not the forces existing within the bodies. As a result,
field quantities existing in the air gap only are considered in dis-
cussing the transfer of momentum.

Since the linear momentum and angular momentum are conserved
independently, in general, the electromagnetic field will have an
angular momentum with respect to certain point or axis. Take the
moment of equation (1.8) such that

$$\vec{r} \times \text{div } M = \vec{r} \times \vec{f} + \frac{\partial}{\partial t} \vec{r} \times (\varepsilon_o \vec{E} \times \vec{B})$$

...1.12

where $\vec{r}$ is the radius vector from the arbitrary origin to the field
point. Integration over the volume concerned yields

$$\int_V \vec{r} \times \text{div } M dV = \int_V \vec{r} \times \vec{f} dV + \frac{\partial}{\partial t} \int_V \vec{r} \times (\varepsilon_o \vec{E} \times \vec{B}) dV$$

...1.13

Again $\vec{r} \times \text{div } M \, dV$ has the important property that the integral
over any volume can be transformed into an integral over the surface
S enclosing the volume such that

$$\int_V \vec{r} \times \text{div } M \, dV = \int_S \vec{r} \times (M.\hat{n}) dS \qquad \ldots 1.14$$

Hence

$$\int_S \vec{r} \times (M.\hat{n}) dS = \int_V \vec{r} \times \vec{f} \, dV + \frac{\partial}{\partial t} \int_V \vec{r} \times (\epsilon_0 \vec{E} \times \vec{B}) dV \qquad \ldots 1.15$$

Equation (1.15) expresses the conservation of angular momentum in an electromagnetic system. The left-hand side is the total inward flux of electromagnetic angular momentum through the surface. The first term on the R.H.S. gives the mechanical torque produced, that is, the rate of conversion of electromagnetic angular momentum to mechanical angular momentum. The second term may be interpreted as the rate at which the electromagnetic angular momentum stored within the volume is changing. As in the case of linear momentum,

$$g_{e\omega} = \vec{r} \times (\epsilon_0 \vec{E} \times \vec{B})$$

is taken as the electromagnetic angular momentum density. From equation (1.11A).

$$g_{e\omega} = \vec{r} \times \mu_0 \epsilon_0 (\vec{E} \times \vec{H})$$

The flux density of the angular momentum of a circularly polarized wave about the axis of propagation may be shown to be P/ω, where P is the power flow per unit area and ω is the angular frequency. This relation may also be derived from a simple quantum argument. Each photon comprising the beam has an angular momentum $\pm h/2\pi$, where h is Planck's constant. If w is the energy density, the total number of photons per unit volume is $n = w/hf = P/chf$, where f is the frequency of radiation. If the spins of all the photons are aligned

in the positive sense, the total angular momentum per unit volume is

$$g_{em} = \frac{P}{chf} \cdot \frac{h}{2\pi} = \frac{P}{\omega c}$$

This momentum travels at the velocity of light and hence the flux density of angular momentum crossing unit area normal to the direction of propagation is equal to $\frac{Pc}{\omega c} = \frac{P}{\omega}$. If the waves are absorbed by a screen disposed perpendicularly to the direction of propagation, the torque per unit area would be $\frac{P}{\omega}$.

1.5 Electromagnetic Torque on a Body in Free Space

In an electromagnetic system if a body is surrounded by free space (air), then

$$\int_S \vec{r} \times (M.\vec{n}) dS$$

gives uniquely the total inward flow of electromagnetic angular momentum. This has been discussed in the previous section.

Since for sinusoidal steady state the time averages of

$$\frac{\partial}{\partial t} \int_V \vec{r} \times (\epsilon_o \vec{E} \times \vec{B}) dV \quad \text{and} \quad \frac{\partial}{\partial t} \int_V \vec{r} \times \mu_o \epsilon_o (\vec{E} \times \vec{H}) dV$$

are zero, the mechanical torque of electromagnetic origin on the body can be obtained uniquely from

$$T_{em} = \int_S \vec{r} \times (M.\vec{n}) dS \qquad \qquad \ldots 1.16$$
$$\text{(S in free space)}$$

For the transient state, the full equation

$$\int_V \vec{r} \times \vec{f} dV = \int_S \vec{r} \times (M.\vec{n}) dS - \frac{\partial}{\partial t} \int_V g_{e\omega} dV \qquad \qquad \ldots 1.17$$

has to be used.

If for physical reasons the electric and magnetic field intensiities inside the body are vanishingly small, the momentum storage inside it is negligible. For such a body the mechanical torque for both steady and transient state is given by

$$T_{em} = \int_S \vec{r} \times (M.\vec{n}) dS - \frac{\partial}{\partial t} \mu_0 \epsilon_0 \int_{V-V_0} \vec{r} \times (\vec{E} \times \vec{H}) dV \qquad \ldots 1.17A$$

where V = the volume enclosed by S

V_0 = volume of the body

This equation will be applicable in an induction machine (in fact in any machine) with the rotor having iron of very high permeability.

CHAPTER 2

THE MACHINE MODEL AND THE

BOUNDARY VALUE PROBLEM

2.1 Introduction

Field theory analysis of electrical machines involves solutions
of appropriate differential equations derived from Maxwell's equa-
tions for specified conditions. In the mathematical sense it is a
boundary value problem with due consideration of the initial condi-
tions for transient analysis. Normal machine geometry is so
complicated that exact analysis of the electromagnetic field problem
is highly complex and sometimes impracticable. This requires the
reduction of the actual machine into a simpler mathematical model
with some appropriate and reasonable assumptions.

2.2 Cylindrical Model

In the simplest and quite common machine model, the inner
surface of the stator and the outer surface of the rotor are concen-
tric cylinders, separated by a small air gap. On these surfaces
are slots, parallel to or slightly skewed, in which lie the current
carrying conductors. The distribution of the conductors around the
periphery of the machine is such that it forms a periodic function.
The conductors belonging to the different phases of the same mag-
netic member are similar, but displaced from each other by suitable
angles. The stator and rotor may have similar winding arrangements
(wound rotor machines). Quite often squirrel cage windings are used
on the rotor. Sometimes a continuous conducting element, rather than

a network of individual bars is used on the rotor. Such a rotor may

be formed by plating, which will provide a conducting layer on the

surface. This machine may be called a sleeve rotor machine.

2.3 Reference Frame

Because of the cylindrical symmetry it is suitable to use

cylindrical co-ordinates (r,θ,z) for the analysis of such a machine.

Furthermore it is convenient to use reference frames fixed to each

magnetic member such that the stator reference system is stationary

with respect to stator and rotor reference system is stationary

with respect to rotor, but rotates in space with respect to the

fixed reference frame of the stator. The velocities involved are

very small compared to that of light. This makes time and length

measure in the two systems approximately the same (no relativistic

time and length difference).
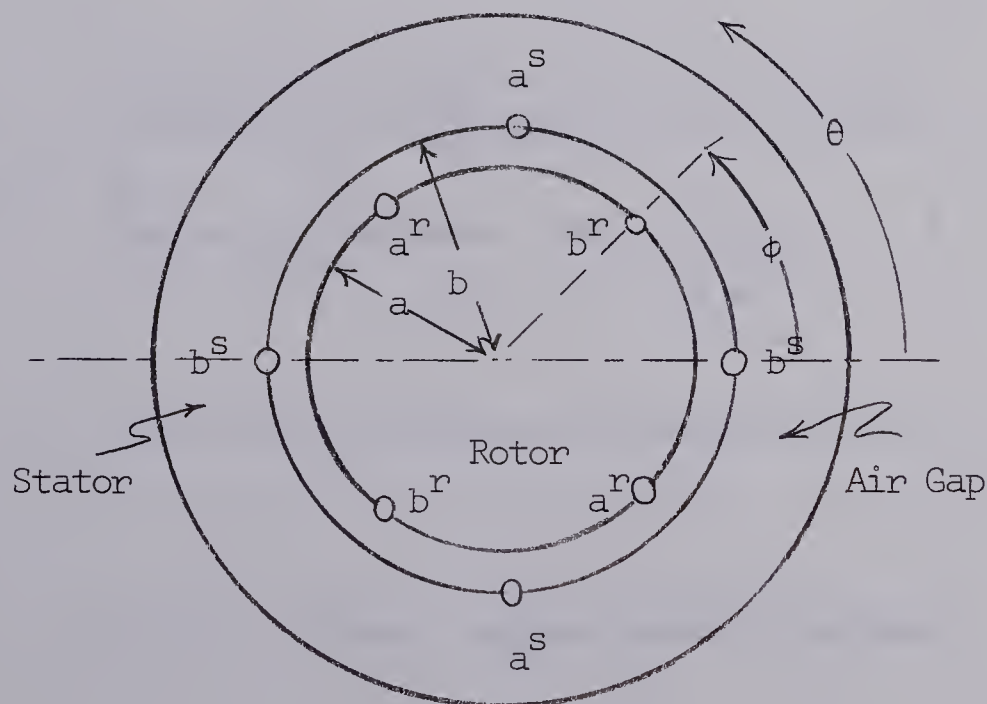
Fig. 2.1 - Schematic Diagram of a Wound Rotor Machine

Fig. (2.1) shows the schematic diagram of a two phase wound rotor machine. Only the centre turns of each phase are shown. ϕ is the space angle between the axes of stator phase (a) and rotor phase (a). When the rotor rotates ϕ is a function of time. Therefore the effect of rotation must be taken into account when quantities of one frame are referred to another.

2.4 Reduction to Boundary Value Problem

Consider Fig. (2.2) where regions (1), (2) and (3) represent respectively air, a conducting layer with continuous current distribution and iron.

Region (3) – Iron
$\mu_r \rightarrow \infty$
$\vec{H} \rightarrow 0$
$\vec{E} \rightarrow 0$

Region (2)–Conductor
$\mu = \mu_0$
$\delta = \delta$

Region (1)–Air
$\mu = \mu_0$
$\delta = 0$
$\varepsilon = \varepsilon_0$
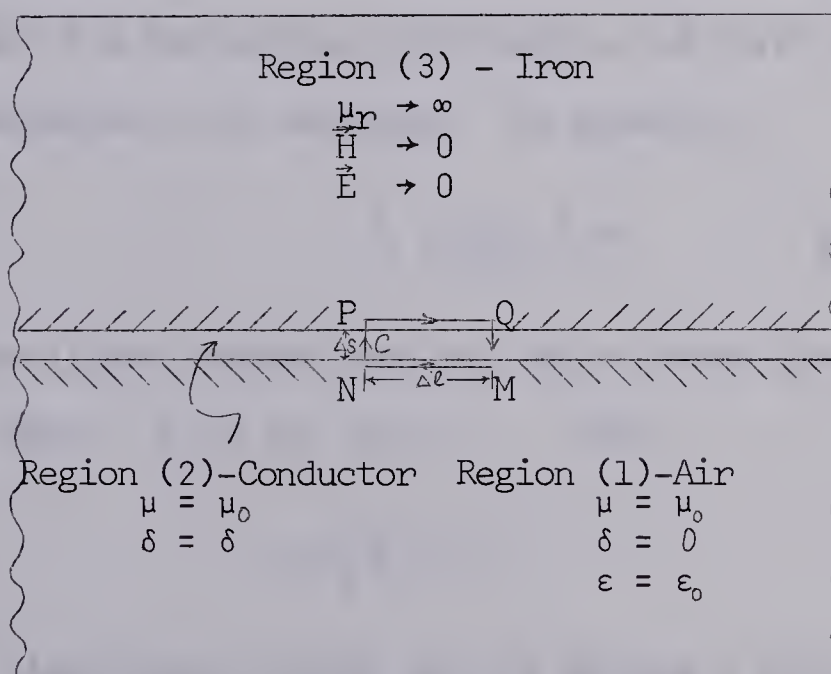
Fig. 2.2 – Three – Region Boundary Surfaces

The sides MN and PQ, each of length $\Delta\ell$, of the contour C lie in either face of the conducting layer and the sides NP and QM penetrating the layer are of length Δs each. Integrate

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

over the surface bounded by the contour and apply Stokes' theorem to obtain

$$\int_C \vec{H} \cdot \vec{d\ell} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{dS} + \int_S \vec{J}_f \cdot \vec{dS} \qquad \ldots 2.1$$

From the above equation it may be shown[40] that in the limit as $\Delta s \to 0$, i.e. region (2) vanishes to a surface

$$\hat{n} \times (\vec{H}_3 - \vec{H}_1) = \lim_{\Delta s \to 0} \left(\frac{\partial \vec{D}}{\partial t} + \vec{J}_f \right) \Delta s \qquad \ldots 2.2$$

Because D and its derivatives are bounded, the first term on the R.H.S. of equation (2.2) vanishes. The quantity

$$\vec{I}_1 = \lim_{\Delta s \to 0} \vec{J}_f \, \Delta s \qquad \text{amp/m} \qquad \ldots 2.3$$

known as the linear current density, can be other than zero on the assumption that $\vec{J}_f \to \infty$ as $\Delta s \to 0$. Thus

$$\hat{n} \times (\vec{H}_3 - \vec{H}_1) = \vec{I}_1 \qquad \ldots 2.4$$

For a distributed winding let the contour C with sides NP and QM in the tooth regions enclose the κth slot. Also let there be N_κ conductors in this slot each carrying a current i_κ. The total current flowing in the slot is then $i_\kappa N_\kappa$. In view of the fact that

$\mu_r \to \infty$, the magnetic field intensity along NP, PQ and QM is zero. Hence, the line integral of $\vec{H}$ taken around the slot is $H_1 \Delta l$, which from Maxwell's equations is equal to the current in the slot. Thus, $\vec{H}_1 \Delta \ell = N_k i_k$. If the number of turns per unit length is $N_{ik} = N_k / \Delta \ell$, it follows that $H_1 = N_{1k} i_k$. Clearly $i_1 = N_{1k} i_k$ acts as a linear surface current density.

For the distributed winding, the air gap surface may now be viewed as consisting of a series of a finite number of discontinuous linear current density pulses. Due to the periodic distribution of these pulses, it is possible to expand them into Fourier space harmonics[33]. For the continuous winding the linear current density would be a continuous periodic function. In fact any practical winding including the effects of skewing may be handled in this manner[33].

The electromagnetic system of the machine is thus reduced to a boundary value problem with air gap surfaces represented by Fourier series of linear current density. Inside iron $\vec{H} \to 0$, and in crossing the surface the tangential component of the magnetic field changes discontinuously by an amount equal to the linear current density. The other boundary condition may be shown to be that the normal component of the magnetic flux density must be continuous across any surface of discontinuity. This follows from the fact that $\vec{\nabla} . \vec{B} = 0$.

2.5 The Two-Phase Model to be Analysed and Summary of

   Basic Assumptions

   (a) A two pole machine with two stator windings placed in
        space quadrature will be considered. It will be taken

that the conductors forming a pole are similar to
those forming the other pole so that the linear current
density will be given by the odd harmonics,

$$i_{1na}^S = \sum_{n=1,3,\ldots}^{\infty} N_n^S \, i_a^S \, \sin n\theta^S \qquad \ldots 2.5$$

$$i_{1nb}^S = \sum_{n=1,3,\ldots}^{\infty} N_{-n}^S \, i_b^S \, \sin n \, (\theta^S - \pi/2) \qquad \ldots 2.6$$

where i_a^S and i_b^S are currents in stator phase (a) and (b)
respectively. It may be shown that N_n^S is a constant
corresponding to each harmonic, and it takes care of the
space distribution of the conductors.

(b)   The rotor will have either two phase windings similar to
those of the stator or a sleeve rotor winding.

(c)   The machine axial length is considerably larger than its
radius. (This will make the effect of the currents flow-
ing in the end wires negligible.)

(d)   The relative permeability of iron is taken to be infin-
itely larger than that of air.

CHAPTER 3

VECTOR WAVE EQUATION

AND BOUNDARY CONDITIONS

3.1 Introduction

Exact solution of Maxwell's equations are required to establish the concept of electromagnetic angular momentum flow in an induction machine. The solutions may be attempted by defining suitable pot- ential functions. Because of the assumed linearity of the system superposition will be applicable. It has been stated before that each reference system is stationary with respect to the phase winding producing the field. Hence, the mathematical form of the field equations and their solutions for each phase will be similar.

3.2 The Air Gap Vector Potential

In appendix (2) Maxwell's equations are solved in terms of suitably defined potential functions. It follows that the air gap (which is both current and charge free) fields may be obtained from the solution of

$$\nabla^2 \vec{A} - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}}{\partial t^2} = o \qquad \dots 3.1$$

$$\vec{B} = \nabla \times \vec{A} \qquad \dots 3.2$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \qquad \dots 3.3$$

$$\nabla . \vec{A} = o \qquad \dots 3.4$$

The function $\vec{A}$ is the vector potential of the field and can be

uniquely determined by satisfying the boundary conditions. The

use of the potential function $\vec{A}$ simplifies the exact solution of

the field problem by cutting the algebra because all the com-

ponents of the field vectors are included in it.

In cylindrical co-ordinates one may write equations (3.1) and

(3.4) as

$$\nabla^2 \vec{A} = \vec{1}_r \left(\nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} \right) + \vec{1}_\theta \left(\nabla^2 A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \right)$$

$$+ \vec{1}_z \nabla^2 A_z = \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\left. \right\} \ldots 3.5$$

$$\nabla \cdot \vec{A} = \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad \ldots 3.6$$

(1) The machine length has been assumed to be considerably larger

than its radius, thus making the effects of the currents flowing

in the end wires negligible. (As a matter of fact, some fringing

of the fields must occur at the ends of the machine and this may be

neglected unless the length is rather short.)

(2) Furthermore, it is now postulated that the surface current

density is independent of the z-co-ordinate of the system (that is,

current is the same in the entire path in the z-direction and,

also there is no retardation in the z-direction. In a real machine

this is approximately equivalent to neglecting interturn cap-

acitances and leakages between conductors of the same member.)

The above two conditions give

$$\frac{\partial A_z}{\partial z} = 0$$

and taking $A_r = A_\theta = 0$, the divergence condition, namely equation

$\nabla \cdot \vec{A} = 0$ is satisfied. The wave equation is thus reduced to

$$\nabla^2 A_z(r,\theta,t) = \mu_o \epsilon_o \frac{\partial^2 A_z}{\partial t^2}(r,\theta,t) \qquad \ldots 3.7$$

and this has to be solved for the specified boundary conditions.

3.3 Boundary Conditions

Since $\vec{B} = \nabla \times \vec{A}$ and $\vec{A}$ has only a z-component

$$\vec{B} = \vec{i}_1 \frac{1}{r} \frac{\partial A_z}{\partial \theta} + \vec{i}_\theta \left(- \frac{\partial A_z}{\partial r}\right)$$

The boundary conditions discussed in section (2.4) may be

reduced to

(1) at the stator and rotor surfaces

(r = b and r = a)
$$\frac{\partial A_z}{\partial \theta} \quad \text{is continuous}$$

(2) at the stator surface

$$\left.\frac{\partial A_z}{\partial r}\right|_{r=b} = \mu_o i_{in}^s \qquad \ldots 3.8$$

and at the rotor surface

$$\ldots 3.9$$
$$\left.\frac{\partial A_z}{\partial r}\right|_{r=a} = - \mu_o i_{in}^r$$

CHAPTER 4

SOLUTION OF THE WAVE EQUATION

IN OPERATIONAL FORM

4.1 Introduction

In general, the solution of the three dimensional wave equation

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \qquad \qquad \ldots 3.1$$

is needed for the analysis of the machine especially under the

transient condition. However, for the model chosen, if the current

density distribution is independent of the z-co-ordinate of the

system even under the transient condition, the solution of equation

$$\nabla^2 A_z(r,\theta,t) = \mu_0 \epsilon_0 \frac{\partial^2 A_z}{\partial t^2}(r,\theta,t) \qquad \qquad \ldots 3.7$$

will be sufficient for the study of machine performance. (From now

on $A_z(r,\theta,t)$ will be referred to as A). An equation similar to

(3.7) is valid for each of the phases of the stator and rotor with

the reference system being stationary with respect to the winding

concerned.

The wave equation may be solved by various methods[41]. For

ordinary steady state analysis of the machine, a classical approach

is to seek cylindrical wave functions, which satisfy equation (3.7)

subject to the boundary conditions of section (3.3). In such a

steady state solution, the frequency is equal to that of the driving

function (surface current density). In the transient case the

initial conditions must be considered and the wave equation may

be solved as an eigenvalue problem[42]. Imposing the boundary

conditions upon a solution obtained by Bernoulli's separation method, the eigenvalues and the corresponding eigenfunctions of the problem may be determined. To satisfy the initial conditions it is necessary to be able to express an arbitrary function as an infinite series of the eigenfunctions.

By using the method of the Laplace Transform, an operational solution of the wave equation (3.7) is shown in this chapter. The general solution thus obtained can be conveniently applied for certain analysis of the machine problems in both the steady and transient states (see chapters 5, 8 and 9). Also, the general solution may be approximated to define frequency independent parameter for any transient and steady state analysis of the machine from a conventional quasi-static point of view (see chapter 7).

4.2 The Laplacian Transformation

The Laplace transforms are a pair of functions related by two fundamental equations of the Operational Calculus:

(a) the Laplace transformation

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

and (b) by the Fourier - Mellin theorem, the inverse formula

$$f(t) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s) e^{st} ds \qquad t \geqslant 0$$

The integrals are supposed to converge, with t limited to real values, while s is, in general, a complex number. The convergence[43] of the first integral is assured if,

$$\int_0^\infty |f(t)e^{-st}|dt = \int_0^\infty |f(t)|e^{-Re(s).t}dt$$

converges. The line integral in the inversion formula is evaluated by transformation of the path of integration to a suitable closed contour and the use of the Calculus of Residues. The inverse transformation is a solution of the Laplace transformation and vice versa, so that the functions related by one of the above equations are of necessity related by the other also.

4.3 The Operational Equation

Expand equation (3.7) as

$$\nabla^2 A = \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \qquad \ldots 4.1$$

where

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}}$$

The solution of this equation satisfying two boundary and two initial conditions, is sought. The initial conditions are

$$\underset{t\to o}{\text{Lim}} \; A(r,\theta,t) = A_o(r,\theta) = A_o$$

$$\underset{t\to o}{\text{Lim}} \; \frac{\partial A}{\partial t}(r,\theta,t) = A_1(r,\theta) = A_1$$

Take the Laplace transform of (4.1) assuming that the integrals

$$\int_o^\infty Ae^{-st}dt, \quad \int_o^\infty \frac{\partial A}{\partial t} e^{-st}dt, \text{ etc.}$$

for s>o, exist and the operations of differentiating with respect to the spatial co-ordinates and taking the Laplace transform can be interchanged such that

$$\int_0^\infty e^{-st} \nabla^2 A \, dt = \nabla^2 \int_0^\infty A e^{-st} dt .$$

Let

$$\bar{A} = \int_0^\infty A e^{-st} dt = \bar{A}(r,\theta,s)$$

Then

$$\int_0^\infty \frac{\partial A}{\partial t} e^{-st} dt = \left. A e^{-st} \right]_0^\infty + s \int_0^\infty A e^{-st} dt$$

$$= -A_0 + s\bar{A}$$

and

$$\int_0^\infty \frac{\partial^2 A}{\partial t^2} e^{-st} dt = \left. \frac{\partial A}{\partial t} e^{-st} \right]_0^\infty + s \int_0^\infty \frac{\partial A}{\partial t} e^{-st} dt$$

$$= -(s A_0 + A_1) + s^2 \bar{A}$$

Therefore, with the above assumptions as to the nature of the vector potential A, one obtains from equation (4.1) and the initial conditions the operational equation in the s-domain as

$$\frac{\partial^2 \bar{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{A}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{A}}{\partial \theta^2} - \frac{s^2 \bar{A}}{c^2} = -\frac{1}{c^2}(s A_0 + A_1) \qquad \ldots 4.2$$

This is the two dimensional form of the Helmholtz Equation and has to be solved to satisfy the boundary conditions.

4.4 Solution of Helmholtz Equation

For the complementary function write

$$\frac{\partial^2 \overline{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{\Psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \overline{\Psi}}{\partial \theta^2} - \frac{s\overline{\Psi}}{c^2} = 0 \qquad \dots 4.3$$

Also, let

$$\overline{\Psi}(r,\theta,s) = F_1(r,s)\, F_2(\theta) \qquad \dots 4.4$$

Substitution of equation (4.4) in equation (4.3) and a few simple manipulations yield

$$\frac{1}{F_1}\frac{dF_1}{dr^2} + \frac{1}{rF_1}\frac{dF_1}{dr} + \frac{1}{r^2 F_2}\frac{d^2 F_2}{d\theta^2} - \frac{s^2}{c^2} = 0 \qquad \dots 4.5$$

Let

$$\frac{d^2 F_2}{F_2 d\theta^2} = -n^2$$

From which,

$$F_2 = C_n' \sin(n\theta + \beta_n) \qquad \dots 4.6$$

(n being the order of the space harmonic).

Substitute this in equation (4.5) to obtain

$$\frac{d^2 F_1}{dr^2} + \frac{dF_1}{r\,dr} - \left(\frac{s^2}{c^2} + \frac{n^2}{r^2}\right)F_1 = 0 \qquad \dots 4.7$$

This is the Modified Bessel Equation and its solution is

$$F_1 = A'_n I_n \left(\frac{s}{c}r\right) + B'_n K_n \left(\frac{s}{c}r\right)$$

where $I_n\left(\frac{s}{c}r\right)$ and $K_n\left(\frac{s}{c}r\right)$ denotes the Modified Bessel functions of the first and second kind respectively, both of order n and argument $\frac{s}{c}r$.

Put

$$\frac{s}{c} = k, \quad C_n = A'_n C'_n \quad \text{and} \quad D_n = B'_n C'_n$$

Hence, the complementary function is

$$\psi(r,\theta,s) = \quad C_n I_n(kr) + D_n K_n(kr) \quad \sin(n\theta + \beta_n)$$

$$\ldots 4.8$$

where C_n, D_n and B_n are arbitrary constants to be determined from the boundary conditions.

For the particular integral corresponding to the right hand side of equation (4.2), Lagrangian method of variation of parameters[44] may be used. However, it is more directly obtained as follows:

Assume that

(A) $A_o(r,\theta) = R_o \dfrac{\left[J'_n(mx)Y_n(mr) - Y'_n(mx)J_n(mr)\right]}{\left[J'_n(ma)Y'_n(mb) - Y'_n(ma)J'_n(mb)\right]} \sin(n\theta + \beta_n)$

and

(B) $A_1(r,\theta) = R_1 \dfrac{\left[J'_n(mx)Y_n(mr) - Y'_n(mx)J_n(mr)\right]}{\left[J'_n(ma)Y'_n(mb) - Y'_n(ma)J'_n(mb)\right]} \sin(n\theta + \beta_n)$

where, $J_n(mr)$ and $Y_n(mr)$ are the Bessel functions of the first and second kind respectively of order $n, R_o, R, m = \dfrac{\omega}{c}$ (ω being angular frequency) are constants and

x = a or b (constants). In chapter 8 it will be shown that the steady state solution of the vector potential due to the stator currents will contain a factor similar to (A) with x = a and, that due to the rotor currents will contain a similar factor with x = b. The assumption of the particular form of the initial conditions $A_0(r,\theta)$ and $A_1(r,\theta)$, given by (A) and (B) respectively, implies that the transients will be created by a sudden change when the machine is operating at the steady state with a single frequency current in the phase concerned.

Let the particular solution sought be of the type

$$\overline{A}_P = \overline{R}_n \sin(n\theta + \beta_n)$$

(P denoting particular integral)

Substitution into (4.2) yields

$$\frac{d^2\overline{R}_n}{dr^2} + \frac{d\overline{R}_n}{rdr} - \left(K^2 + \frac{n^2}{r^2}\right)\overline{R}_n$$

$$= -\left(\frac{sR_0 + R_1}{c^2}\right)\cdot\frac{\left[J_n'(mx)\,Y_n'(mr) - Y_n'(mx)\,J_n(mr)\right]}{\left[J_n'(ma)\,Y_n'(mb) - Y_n'(ma)\,J_n'(mb)\right]}$$

$$\ldots 4.9$$

Adding and subtracting $m^2\overline{R}_n$, the above equation may be reduced to

$$\frac{d^2\overline{R}_n}{dr^2} + \frac{d\overline{R}_n}{rdr} + \left(m^2 - \frac{n^2}{r^2}\right)\overline{R}_n - (x^2 + m^2)\overline{R}_n$$

$$= -\left(\frac{sR_0 + R_1}{c^2}\right)\cdot\frac{\left[J_n'(mx)\,Y_n(mr) - Y_n'(mx)\,J_n(mr)\right]}{\left[J_n'(ma)\,Y_n'(mb) - Y_n'(ma)\,J_n'(mb)\right]}$$

$$\ldots 4.10$$

Since $Y_n(mr)$ and $J_n(mr)$ are the solutions of the Bessel Equation of order n,

$$\overline{R}_n = \frac{(sR_o+R_1)}{c^2(k^2+m^2)} \cdot \frac{\left[J_n'(mx)Y_n(mr)-Y_n'(mx)J_n(mr)\right]}{\left[J_n'(ma)Y_n'(mb)-Y_n'(ma)J_n'(mb)\right]}$$

...4.11

The solution of equation (4.2) for the nth space harmonic is, therefore, given by

$$\overline{A}_n(r,\theta,s) = \left[C_n I_n(kr) + D_n K_n(kr)\right] \sin(n\theta + \beta_n)$$

$$+ \frac{(sR_o+R_1)}{c^2(k^2+m^2)} \cdot \frac{\left[J_n'(mx)Y_n(mr)-Y_n'(mx)J_n(mr)\right]}{\left[J_n'(ma)Y_n'(mb)-Y_n'(ma)J_n'(mb)\right]} \sin(n\theta+\beta_n)$$

...4.12

The general solution for $\overline{A}$ is the summation of $\overline{A}_n$ over n such that

$$\overline{A}(r,\theta,s) = \sum_{n=1}^{\infty} \overline{A}_n(r,\theta,s) \qquad \qquad ...4.12A$$

As stated before, the initial conditions introduced in obtaining equations (4.12) and (4.12A) correspond to a single time frequency ω. If more than one time function (conceivably a Fourier series of time harmonics) is involved, there would be initial conditions similar to (A) and (B) for each of the time frequencies present. Each set of the initial conditions would give a corresponding particular integral and the most general solution of equation (4.2) to include all such cases may be expressed as

$$\overline{A}(r,\theta,s) = \sum_{\nu} \sum_{n=1}^{\infty} \overline{A}_{\nu,n}(r,\theta,s) \qquad \qquad ...4.12B$$

where, ν is the order of the time function (harmonic in the case of Fourier Series). However, in this analysis only one set of initial conditions ($\nu = 1$) will be considered.

4.5 Boundary Conditions and Vector Potentials due to the Phase Currents

The boundary conditions in the s-domain are

(1) at the surface of excitation the change in the tangential component of the magnetic field intensity in the s-domain is equal to the linear current density in the same domain.

(2) at the other surface the tangential component of the magnetic field intensity vanishes.

It is convenient to use superposition and hence the vector potential corresponding to each phase current will be determined.

(A) Stator Phase (a): For the stator phase (a),

$$\frac{\partial \bar{A}^s_{an}}{\partial r}\bigg|_{r=b} = \mu_0 N^s_n \bar{I}^s_a(s)$$

(NOTE: The transform of the linear current density (amp/m)has been replaced by that of the actual current (amp) multiplied by a constant. See section 2.5.)

$$\frac{\partial \bar{A}^s_{an}}{\partial r}\bigg|_{r=a} = 0$$

$\theta = \theta^s$

$X = a$

$\beta_n = 0$

$m_s = \dfrac{\omega_s}{c}$

$R_0 = \dfrac{\mu_0 N^s_n I^s R^s_{0a}}{m_s}$

$R_1 = \dfrac{\mu_0 N^s_n I^s R^s_{1a}}{m_s}$

where, R_{oa}^S and R_{1a}^S are constants and ω_s is the angular frequency of the stator currents.

Using the above relations with equation (4.12) one obtains

$$C_{an}^S = -D_{an}^S \frac{K_n'(\kappa a)}{I_n'(\kappa a)}$$

and

$$\kappa \left[C_{an}^S I_n'(\kappa b) + D_{an}^S K_n'(\kappa b) \right] + \frac{\mu_0 N_n^S I^S (s R_{oa}^S + R_{1a}^S)}{c^2 (\kappa^2 + m_s^2)}$$

Let

$$= \mu_0 N_n^S \overline{I}_a^S (s)$$

$$\overline{I}_a^S = \left[\mu_0 N_n^S \overline{I}_a^S(s) - \frac{\mu_0 N_n^S I^S (s R_{oa}^S + R_{1a}^S)}{c^2(\kappa^2 + m_s^2)} \right]$$

Then

$$D_{an}^S = \frac{1}{\kappa} \frac{I_n'(\kappa a) \overline{I}_a^S}{\left[K_n'(\kappa b) I_n'(\kappa a) - I_{1n}'(\kappa b) K_n'(\kappa a) \right]}$$

and

$$C_{an}^S = -\frac{1}{\kappa} \frac{K_n'(\kappa a) \overline{I}_a^S}{\left[K_n'(\kappa b) I_n'(\kappa a) - I_n'(\kappa b) K_n'(\kappa a) \right]}$$

Hence,

$$\overline{A}_{an}^S (r \, \theta \, s) = \frac{1}{\kappa} \frac{\left[K_n'(\kappa a) I_n(\kappa r) - I_n'(\kappa a) K_n(\kappa r) \right]}{\left[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b) \right]} \overline{I}_a^S \sin n\theta^S$$

$$+ \frac{\mu_0 N_n^S I^S}{m_s} \cdot \frac{s R_{oa}^S + R_{1a}^S}{c^2(\kappa^2 + m_s^2)} \frac{\left[J_n'(m_s a) Y_n(m_s r) - Y_n'(m_s a) J_n(m_s r) \right]}{\left[J_n'(m_s a) Y_n'(m_s b) - Y_n'(m_s a) J_n'(m_s b) \right]} \times$$

$$\left[\sin n\theta^S \right] \qquad \qquad \dots 4.13$$

and

$$\bar{A}_a^S (r, \theta, s) = \sum_{n=1,3,\cdots}^{\infty} \bar{A}_{an}^S (r, \theta, s) \qquad \ldots 4.13A$$

(B) Stator Phase (b): Following the same procedure as for the stator phase (a), the vector potential for the stator phase (b) is obtained such that,

$$\bar{A}_{bn}^S (r, \theta, s) = \frac{1}{k} \frac{[K_n'(\kappa a) I_n (\kappa r) - I_n'(\kappa a) K_n(\kappa r)]}{[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b)]} \bar{I}_b^S \sin n (\theta^S - \pi/2)$$

$$+ \frac{\mu_0 N_n^S I^S}{m_S} \frac{(s R_{ob}^S + R_{1b}^S)}{c^2 (\kappa^2 + m_S^2)} \frac{[J_n'(m_s a) Y_n (m a) - Y_n'(m_s a) J_n(m_s r)]}{[J_n'(m_s a) Y_n'(m_s b) - Y_n'(m_s a) J_n'(m_s b)]} \sin n (\theta^S - \pi/2)$$

$$\ldots 4.14$$

where,

$$\bar{I}_b^S = \left[\mu_0 N_n^S \bar{I}_b^S (s) - \frac{\mu_0 N_n^S I^S}{c^2} \frac{(s R_{ob}^S + R_{1b}^S)}{(\kappa^2 + m_S^2)} \right]$$

Summing over n

$$\bar{A}_b^S (r, \theta, s) = \sum_{n=1,3,\cdots}^{\infty} \bar{A}_{bn}^S (r, \theta, s) \qquad \ldots 4.14A$$

(C) Rotor Phase (a): The boundary conditions are

$$\frac{\partial \bar{A}_{an}^r}{\partial r} \bigg|_{r=b} = 0$$

$$\frac{\partial \bar{A}_{an}^r}{\partial r} \bigg|_{r=a} = -\mu_0 N_n^r \bar{I}_a^r (s)$$

Also, $\theta = \theta^r$

$x = b$

$\beta_n = 0$

$m_r = \dfrac{\omega_r}{c}$

$R_o = \dfrac{\mu_o N_n^r R_{oa}^r}{m_r}$

$R_1 = \dfrac{\mu_o N_n^r R_{la}^r}{m_r}$

where, R_{oa}^r and R_{la}^r are constants and ω_r is the angular frequency of the rotor currents.

Substitution of these conditions into equation (4.12) yields,

$$C_{an}^r = -D_{an}^r \frac{K'_n(kb)}{I'_n(kb)}$$

and,

$$k\left[C_{an}^r I'_n(ka) + D_{an}^r K'_n(ka)\right] - \frac{\mu_o N_n^r I^r(sR_{oa}^r + R_{la}^r)}{c^2(k^2+m_r^2)}$$

$$= -\mu_o N_n^r \bar{I}_a^r(s)$$

Let

$$\bar{I}_a^r = \frac{\mu_o N_n^r I^r(sR_{oa}^r + R_{la}^r)}{c^2(k^2 + m_r^2)} - \mu_o N_n^r \bar{I}_a^r(s)$$

Then

$$D_{an}^r = \frac{1}{k}\frac{I'_n(kb)\,\bar{I}_a^r}{\left[K'_n(ka)I'_n(kb)-I'_n(ka)K'_n(kb)\right]}$$

$$C_{an}^r = -\frac{1}{k}\frac{K'_n(kb)\,\bar{I}_a^r}{\left[K'_n(ka)I'_n(kb)-I'_n(ka)K'_n(kb)\right]}$$

Therefore,

$$\bar{A}_{an}^r(r,\theta^r,s) = -\frac{1}{k}\frac{\left[K'_n(kb)I_n(kr)-I'_n(kb)K_n(kr)\right]}{\left[K'_n(ka)I'_n(kb)-I'_n(ka)K'_n(kb)\right]} \bar{I}_a^r \sin n\theta^r$$

$$+ \frac{\mu_o N_n^r I^r}{m_r}\frac{(sR_{oa}^r + R_{la}^r)}{c^2(k^2+m_r^2)}\frac{\left[J'_n(m_r b)Y_n(m_r r)-Y'_n(m_r b)J_n(m_r r)\right]}{\left[J'_n(m_r a)Y'_n(m_r b)-Y'_n(m_r a)J'_n(m_r b)\right]}$$

$$[\sin n\theta^r] \qquad\qquad \ldots 4.15$$

and

$$\bar{A}^r_a(r,\theta^r_s) = \sum_{n=1,3,\cdots}^{\infty} \bar{A}^r_{an}(r,\theta^r_s) \qquad \ldots 4.15A$$

(D) Rotor phase (b): Similar treatment of the rotor phase (b) gives

$$\bar{A}^r_{bn}(r,\theta^r_s) = -\frac{\left[K'_n(\kappa b)I_n(\kappa r) - I'_n(\kappa b)K_n(\kappa a)\right]}{\left[K'_n(\kappa a)I'_n(\kappa b) - I'_n(\kappa a)K'_n(\kappa b)\right]} \bar{I}^r_b \sin n(\theta^r - \pi/2)$$

$$+ \frac{\mu_o N^r_n I^r}{mr} \frac{(sR^r_{ob}+R^r_{1b})}{c^2(\kappa^2+m^2_r)} \frac{\left[J'_n(m_r b)Y_n(m_r r) - Y'_n(m_r b)J_n(m_r r)\right]}{\left[J'_n(m_r a)Y'_n(m_r b) - Y'_n(m_r a)J'_n(m_r b)\right]} \cdot$$

$$\left[\sin n(\theta^r - \pi/2)\right] \qquad \ldots 4.16$$

where

$$\bar{I}^r_b = \left[\frac{\mu_o N^r_n I^r(sR^r_{ob}+R^r_{1a})}{c^2(\kappa^2+m^2_r)} - \mu_o N^r_n \bar{I}^r_a(s)\right]$$

Summation over n yields

$$\bar{A}^r_b(r,\theta^r_s) = \sum_{n=1,3,\cdots}^{\infty} \bar{A}^r_{dn}(r,\theta^r_s) \qquad \ldots 4.16A$$

The resultant air gap vector potential for all the currents acting together is

$$\bar{A} = \sum_{n=1,3,\cdots}^{\infty} \left[\bar{A}^s_{an} + \bar{A}^s_{bn} + \bar{A}^r_{an} + \bar{A}^r_{bn}\right] \qquad \ldots 4.17$$

CHAPTER 5

ON THE INVERSE TRANSFORM

5.1 Introduction

The operational solution of the wave equation

$$\nabla^2 A(r,\theta,t) = \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2}(r,\theta,t) \qquad \ldots 3.7$$

assumed the space co-ordinates r and θ as independent of time

t so that the evaluation of the integral in taking the transform of

A and operations of taking partial derivatives with respect to r and

θ are interchangeable. In equations (4.13) through (4.16) if the

current transforms are known, in principle the inverse transforms

may be calculated in a routine manner. In general each current

transform may be determined from an equation similar to

$$\sigma J_f = E_0 - \frac{\partial A}{\partial t} \qquad \ldots 5.1$$

where σ is the resistivity of the surface material, E_0 is the field

intensity applied from external source and $\frac{\partial A}{\partial t}$ is the induced field.

Equation (5.1) requires the stator vector potentials referred

to the rotor co-ordinate system and vice versa. In determining $\frac{\partial A}{\partial t}$

and its transform, motion of the rotor must be taken into considera-

tion. Equations (4.13) through (4.16) can be easily manipulated to

account for constant rotor speed at sinusoidal steady state (see

chapter 8). However, for the expressions to be obtained in this

chapter, the rotor will be considered as stationary.

5.2 Electric Field Intensities at the Stator and the Rotor Surfaces.

In the time domain

$$E(t) = - \frac{\partial A}{\partial t}$$

Hence, in the s-domain

$$\bar{E}(s) = - s\bar{A} + A_0 \qquad \ldots 5.2$$

Let

$$\bar{X}_n^{ss}(s) = \frac{\left[K_n'(\kappa a) I_n(\kappa b) - I_n'(\kappa a) K_n(\kappa b) \right]}{\left[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b) \right]} \qquad \ldots 5.3$$

$$X_n^{ss}(\omega_s) = \frac{\left[J_n'(m_s a) Y_n(m_s b) - Y_n'(m_s a) J_n(m_s b) \right]}{\left[J_n'(m_s a) Y_n'(m_s b) - Y_n'(m_s a) J_n'(m_s b) \right]} \qquad \ldots 5.4$$

$$\bar{X}_n^{rs}(s) = \frac{\left[K_n'(\kappa a) I_n(\kappa a) - I_n'(\kappa a) K_n(\kappa a) \right]}{\left[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b) \right]} \qquad \ldots 5.5$$

$$X_n^{rs}(\omega_s) = \frac{\left[J_n'(m_s a) Y_n(m_s a) - Y_n'(m_s a) J_n(m_s a) \right]}{\left[J_n'(m_s a) Y_n'(m_s b) - Y_n'(m_s a) J_n'(m_s b) \right]} \qquad \ldots 5.6$$

$$\bar{X}_n^{rr}(s) = \frac{\left[K_n'(\kappa b) I_n(\kappa a) - I_n'(\kappa b) K_n(\kappa a) \right]}{\left[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b) \right]} \qquad \ldots 5.7$$

$$X_n^{rr}(\omega_r) = \frac{\left[J_n'(m_r b) Y_n(m_r a) - Y_n'(m_r b) J_n(m_r a)\right]}{\left[J_n'(m_r a) Y_n'(m_r b) - J_n'(m_r b) Y_n'(m_r a)\right]} \qquad \ldots 5.8$$

$$\bar{X}_n^{sr}(s) = \frac{\left[K_n'(\kappa b) I_n(\kappa b) - I_n'(\kappa b) K_n(\kappa b)\right]}{\left[K_n'(\kappa a) I_n'(\kappa b) - I_n'(\kappa a) K_n'(\kappa b)\right]} \qquad \ldots 5.9$$

$$X_n^{sr}(\omega_r) = \frac{\left[J_n'(m_r b) Y_n(m_r b) - Y_n'(m_r b) J_n(m_r b)\right]}{\left[J_n'(m_r b) Y_n'(m_r b) - Y_n'(m_r a) J_n'(m_r b)\right]} \qquad \ldots 5.10$$

Hence, at the stator surface ($r = b$), the nth harmonic field intensities are

$$\bar{E}_{ar}^{ss} = - s\mu_0 N_n^s C_a^{ss} \sin n\theta^s \qquad \ldots 5.11$$

$$\bar{E}_{bn}^{ss} = - s\mu_0 N_n^s C_b^{ss} \sin n(\theta^s - \pi/2) \qquad \ldots 5.12$$

$$\bar{E}_{an}^{sr} = - s\mu_0 N_n^r C_a^{sr} \sin n\theta^r \qquad \ldots 5.13$$

$$\bar{E}_{bn}^{sr} = - s\mu_0 N_n^r C_b^{sr} \sin n(\theta^r - \pi/2) \qquad \ldots 5.14$$

where,

$$C_a^{SS}(s) = \left[\frac{1}{\kappa} \bar{X}_n^{SS}(s) \bar{I}_a^{S}(s) - \frac{I^{S}}{C^2} \left(\frac{sR_{0a}^{S} + R_{1a}^{S}}{\kappa^2 + m_s^2} \right) \left(\frac{\bar{X}_n^{SS}(s)}{\kappa} - \frac{X_n^{SS}(\omega_s)}{m_s} \right) \right.$$
$$\left. - \frac{I^{S} R_{0a}^{S}}{s\, m_s} X_n^{SS}(\omega_s) \right] \qquad \dots 5.15$$

$$C_b^{SS}(s) = \left[\frac{1}{\kappa} \bar{X}_n^{SS}(s) \bar{I}_b^{S}(s) - \frac{I^{S}}{C^2} \left(\frac{sR_{0b}^{S} + R_{1b}^{S}}{\kappa^2 + m_s^2} \right) \left(\frac{\bar{X}_n^{SS}(s)}{\kappa} - \frac{X_n^{SS}(\omega_s)}{m_s} \right) \right.$$
$$\left. - \frac{I^{S} R_{0b}^{S}}{s\, m_s} X_n^{SS}(\omega_s) \right] \qquad \dots 5.16$$

$$C_a^{Sr}(s) = \left[\frac{1}{\kappa} \bar{X}_n^{Sr}(s) \bar{I}_a^{r}(s) - \frac{I^{r}}{C^2} \left(\frac{sR_{0a}^{r} + R_{1a}^{r}}{\kappa^2 + m_r^2} \right) \left(\frac{\bar{X}_n^{Sr}(s)}{\kappa} - \frac{X_n^{Sr}(\omega_r)}{m_r} \right) \right.$$
$$\left. - \frac{I^{r} R_{0a}^{r}}{s\, m_r} X_n^{Sr}(\omega_r) \right] \qquad \dots 5.17$$

$$C_b^{Sr}(s) = \left[\frac{1}{\kappa} \bar{X}_n^{Sr}(s) \bar{I}_b^{r}(s) - \frac{I^{r}}{C^2} \left(\frac{sR_{0b}^{r} + R_{1b}^{r}}{\kappa^2 + m_r^2} \right) \left(\frac{X_n^{-Sr}(s)}{\kappa} - \frac{X_n^{Sr}(\omega_r)}{m_r} \right) \right.$$
$$\left. - \frac{I^{r} R_{0b}^{r}}{s\, m_r} X_n^{Sr}(\omega_r) \right] \qquad \dots 5.18$$

Similarly at the rotor surface (r = a) the field intensities are

$$\bar{E}_{an}^{rs} = - s\mu_0 N_n^{S} C_a^{rs} \sin n\theta^{S} \qquad \dots 5.19$$

$$\bar{E}_{bn}^{rs} = - s\mu_0 N_n^{S} C_b^{rs} \sin n(\theta^{S} - \pi/2) \qquad \dots 5.20$$

$$\bar{E}_{dn}^{rr} = -s\mu_0 N_n^r C_a^{rr} \sin n(\theta^s - \phi) \qquad \ldots 5.21$$

$$\bar{E}_{bn}^{rr} = -s\mu_0 N_n^r C_b^{rr} \sin n(\theta^s - \phi - \pi/2) \qquad \ldots 5.22$$

where

$$C_a^{rs}(s) = \left[\frac{1}{\kappa}\bar{X}_n^{rs}(s)\,\bar{I}_a^s(s) - \frac{I^s}{C^2}\left(\frac{sR_{0a}^s + R_{1a}^s}{\kappa^2 + m_s^2}\right)\left(\frac{\bar{X}_n^{rs}(s)}{\kappa} - \frac{X_n^{rs}(\omega_s)}{m_s}\right) \right.$$
$$\left. - \frac{I^s R_{0a}^s}{s\,m_s}X_n^{rs}(\omega_s) \right] \qquad \ldots 5.23$$

$$C_b^{rs}(s) = \left[\frac{1}{\kappa}\bar{X}_n^{rs}(s)\,\bar{I}_b^s(s) - \frac{I^s}{C^2}\left(\frac{sR_{0b}^s + R_{1b}^s}{\kappa^2 + m_s^2}\right)\left(\frac{\bar{X}_n^{rs}(s)}{\kappa} - \frac{X_n^{rs}(\omega_s)}{m_s}\right) \right.$$
$$\left. - \frac{I^s R_{0b}^s}{s\,m_s}X_n^{rs}(\omega_s) \right] \qquad \ldots 5.24$$

$$C_a^{rr}(s) = \left[\frac{1}{\kappa}\bar{X}_n^{rr}(s)\,\bar{I}_a^r(s) - \frac{I^r}{C^2}\left(\frac{sR_{0a}^r + R_{1a}^r}{\kappa^2 + m_r^2}\right)\left(\frac{\bar{X}_n^{rr}(s)}{\kappa} - \frac{X_n^{rr}(\omega_r)}{m_r}\right) \right.$$
$$\left. - \frac{I^r R_{0a}^r}{s\,m_r}X_n^{rr}(\omega_r) \right] \qquad \ldots 5.25$$

$$C_b^{rr}(s) = \left[\frac{1}{\kappa} \, \bar{X}_n^{rr}(s) \, \bar{I}_b^r(s) - \frac{I^r}{c^2} \left(\frac{sR_{ob}^r + R_{1k}^r}{\kappa^2 + m_r^2} \right) \left(\bar{X}_n^{rr}(s) - \frac{X_n^{rr}(\omega_r)}{m_r} \right) \right.$$

$$\left. - \frac{I^r R_{ob}^r}{s m_r} X_n^{rr}(\omega_r) \right] \qquad \ldots 5.26$$

5.3 Line Integrals

Note that the resolution of the actual current into Fourier space harmonics of linear current density may be taken as that each of these harmonics is produced by a fictitious winding whose conductors are sinusoidally distributed in space. For every cycle of the fundamental space component there are n cycles of the nth harmonic winding. Let any of these n cycles be called the nth harmonic coil of the nth harmonic winding.

Take the Laplace transform of equation (5.1) and integrate it to obtain

$$\int_{\ell_p} R_{\ell n} \, \bar{I}(s) \, d\ell_p = \int_{v\ell_p} \bar{E}_{on}(s) \, d\ell_p + \int_{\ell_p} \bar{E}_n(s) \, d\ell_p \qquad \ldots 5.27$$

where ℓ_p corresponds to the total path of integration in the nth harmonic coil.

In equation (5.27), $\int_{\ell_p} R_{\ell_n} \bar{I}(s) d\ell_p$ gives the total resistance drop of the nth harmonic coil, $\int_{\ell_p} \bar{E}_{on}(s) d\ell_p$ gives the nth harmonic applied voltage and $\int_{\ell_p} \bar{E}_n(s) d\ell_p$ takes care of the total induced voltages. For the last integral consider the intensities corresponding to a conductor at $n\alpha$ in series with one at $n\alpha + \pi$
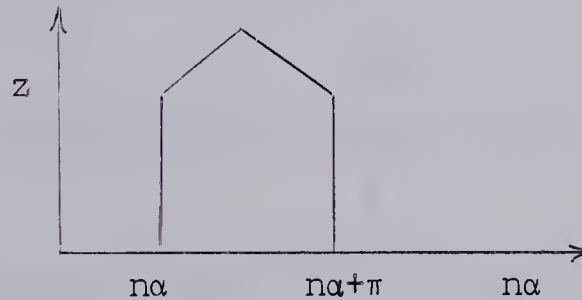
Fig. 5.1 – Harmonic Conductors in Series

Their sum

$$= - s \mu_0 N C \left(\sin n\alpha - \sin(n\alpha + \pi) \right)$$

$$= - 2 s \mu_0 N C \sin n\alpha \qquad \qquad \ldots 5.28$$

Where C is one of the constants defined in the previous section, N and α are dummy variables.

(A) Stator Phase (a): The conductor density is distributed as $N_n^S \sin n\theta^S$. Hence for the nth harmonic coil

$$\int_{\ell_p} \bar{E}_{an}^{ss} d\ell_p = - 2 s \mu_0 N_n^S N_n^S C_a^{ss}(s) \ell b \int_0^{\pi/n} \sin^2 n\theta^S \, d\theta^S$$

$$= - \frac{1}{n} s \mu_0 \pi b \ell N_n^S N_n^S C_a^{ss}(s)$$

$$\int_{l_p} \bar{E}_{bn}^{ss} \, dl_p = -2S\mu_o \, bl \, N_n^S \, N_n^S \, C_b^{ss}(s) \int_0^{\pi/n} \sin n(\theta^S - \pi/2) \sin n\theta^S d\theta^S$$

$$= 0$$

$$\int_{l_p} \bar{E}_{an}^{sr} \, dl_p = -2S\mu_o \, bl \, N_n^S \, N_n^r \, C_a^{sr}(s) \int_0^{\pi/n} \sin n(\theta^S - \phi) \sin n\theta^S d\theta^S$$

$$= -\frac{1}{n} S\mu_o \, \pi bl \, N_n^S \, N_n^r \, C_a^{sr}(s) \cos n\phi \, .$$

$$\int_{l_p} \bar{E}_{bn}^{sr} \, dl_p = -2S\mu_o \, bl \, N_n^S \, N_n^r \, C_b^{sr}(s) \int_0^{\pi/n} \sin n(\theta^S - \phi - \pi/2) \sin n\theta^S d\theta^S$$

$$= -\frac{1}{n} S\mu_o \, \pi \, bl \, N_n^S \, N_n^r \, C_b^{sr}(s) \cos(n\phi + n\pi/2)$$

Hence,

$$\bar{\upsilon}_{an}^{ss}(s) = R_{an}^{ss} \, \bar{I}_a^S(s) + \frac{1}{n} S\mu_o \, \pi bl \, N_n^S \, N_n^S \, C_a^{ss}(s)$$

$$+ \frac{1}{n} S\mu_o \pi bl \, N_n^S \, N_n^r \, C_a^{sr}(s) \cos n\phi$$

$$+ \frac{1}{n} S\mu_o \pi bl \, N_n^S \, N_n^r \, C_b^{sr}(s) \cos(n\phi + n\pi/2) \qquad \ldots 5.29$$

(B) Stator Phase (b): The conductor density is distributed

as $N_n^S \sin n(\theta^S - \pi/2)$ and the integration has to be carried out from $\pi/2$

to $\frac{\pi}{2} + \frac{\pi}{n}$. The result is

$$\bar{v}_{bn}^{ss}(s) = R_{bn}^{ss}\bar{I}_{b}^{s}(s) + \frac{1}{n}\mu_{o}\pi bl\ N_{n}^{s}N_{n}^{s}\ C_{b}^{ss}(s)$$

$$+ \frac{1}{n}s\mu_{o}\pi\ bl\ N_{n}^{s}N_{n}^{r}C_{a}^{sr}(s)\cos\left(n\phi - n\pi/2\right)$$

$$+ \frac{1}{n}s\mu_{o}\pi bl\ N_{n}^{s}N_{n}^{r}C_{b}^{sr}(s)\cos n\phi \qquad \ldots 5.30$$

(C)   Rotor Phase (a):   The conductor density is distributed as $N_{n}^{r}\sin n(\theta^{s}-\phi)$ and the limits of integration are from ϕ to $\phi+\pi/n$. The integration gives

$$\bar{v}_{an}^{ss}(s) = R_{an}^{rr}\bar{I}_{a}^{r}(s) + \frac{1}{n}s\mu_{o}\pi\ al\ N_{n}^{s}N_{n}^{r}C_{a}^{rs}(s)\cos n\phi$$

$$+ \frac{1}{n}s\mu_{o}\pi\ al\ N_{n}^{s}N_{n}^{r}\ C_{b}^{rs}(s)\cos\left(n\phi-n\pi/2\right)$$

$$+ \frac{1}{n}s\mu_{o}\pi\ al\ N_{n}^{r}N_{n}^{r}\ C_{a}^{rr}(s) \qquad \ldots 5.31$$

(D)   Rotor Phase (b):   Similar treatment gives for the phase (b)

$$\bar{v}_{bn}^{rr}(s) = R_{bn}^{rr}\bar{I}_{b}^{r}(s) + \frac{1}{n}s\mu_{o}\pi al\ N_{n}^{s}N_{n}^{r}C_{a}^{rs}(s)\cos\left(n\phi+n\pi/2\right)$$

$$+ \frac{1}{n}s\mu_{o}\pi\ al\ N_{n}^{s}N_{n}^{r}\ C_{b}^{rs}(s)\cos r\phi$$

$$+ \frac{1}{n}s\mu_{o}\pi\ al\ N_{n}^{r}N_{n}^{r}\ C_{b}^{rr}(s) \qquad \ldots 5.32$$

Equations (5.29) through (5.32) are derived for the nth harmonic of a 2-pole 2-phase wound rotor machine with conventional distributed windings on both the members. Simultaneous solutions of these equations would give the current transforms, which may be put back to equations (4.13) through (4.16) for the solutions of the vector potentials in the s-domain. Electromagnetic angular momentum flow at the starting of the rotor (rotor stationary) may be studied for any transient by taking the inverse transforms of the vector potentials. Of course, the transients must be within the assumption made in solving the wave equation.

For the sleeve rotor machine take $\phi = 0$ and write the rotor equations point wise and not in integrated form. This will amount to a change in scale in the induced voltage terms. (The sleeve rotor case is made clear for the fundamental space component in chapter 8 and 9.)

5.4 s-Domain Solution for a Space Harmonic

Consider $\phi = 0$. This will make the (a) phases independent of the (b) phases and in general for this condition the solution will be valid for the sleeve rotor machine.

Let

$$\lambda^S = \frac{1}{n} \mu_o \pi \ell N_n^S N_n^S$$

$$\lambda^r = \frac{1}{n} \mu_o \pi \ell N_n^r N_n^r$$

$$\mathcal{L}^{sr} = \frac{1}{n}\mu_0\pi l\, N_n^s\, N_n^r = \mathcal{L}^{rs}$$

Let

$$\bar{v}_{oan}^{ss}{}^{(s)} = \left[\frac{s\alpha^s b\, I^s(sR_{oa}^s + R_{1a}^s)}{c^2(\kappa^2 + m_s^2)}\left(\frac{x_n^{ss}(\omega_s)}{m_s} - \frac{\bar{x}_n^{ss}(s)}{\kappa}\right) - \frac{\alpha^s b\, I^s R_{oa}^s}{m_s} x_n^{ss}(\omega_s)\right]$$

$$\bar{v}_{oan}^{sr}{}^{(s)} = \left[\frac{s\alpha^{sr} b\, I^r(sR_{oa}^r + R_{1a}^r)}{c^2(\kappa^2 + m_r^2)}\left(\frac{x_n^{sr}(\omega_r)}{m_r} - \frac{\bar{x}_n^{sr}(s)}{\kappa}\right) - \frac{\alpha^{sr} b\, I^r R_{oa}^r}{m_r} x_n^{sr}(\omega_r)\right]$$

$$\bar{v}_{oan}^{rs}{}^{(s)} = \left[\frac{s\alpha^{rs} I^s(sR_{oa}^s + R_{1a}^s)}{c^2(\kappa^2 + m_s^2)}\left(\frac{x_n^{rs}(\omega_s)}{m_s} - \frac{\bar{x}_n^{rs}(s)}{\kappa}\right) - \frac{\alpha^{rs} a\, I^s R_{oa}^s}{m_s} x_n^{rs}(\omega_s)\right]$$

$$\bar{v}_{oan}^{rr}{}^{(s)} = \left[\frac{s\alpha^r a\, I^r(sR_{oa}^r + R_{1a}^r)}{c^2(\kappa^2 + m_r^2)}\left(\frac{x_n^{rr}(\omega_r)}{m_r} - \frac{\bar{x}_n^{rr}(s)}{\kappa}\right) - \frac{\alpha^r a\, I^r R_{oa}^r}{m_r} x_n^{rr}(\omega_r)\right]$$

Also

let $\bar{v}_{an}^{s}(s) = \bar{v}_{an}^{ss}(s) - \bar{v}_{oan}^{ss}(s) - \bar{v}_{oan}^{sr}(s)$

$\bar{v}_{an}^{r}(s) = \bar{v}_{an}^{rr}(s) - \bar{v}_{oan}^{rs}(s) - \bar{v}_{oan}^{rr}(s)$

Hence, for the (a) phases

$$
\begin{bmatrix} \bar{v}_{an}^{s}(s) \\ \\ \bar{v}_{an}^{r}(s) \end{bmatrix} = \begin{bmatrix} R_{an}^{ss} + c\alpha^{s}b\bar{X}_{n}^{ss}(s) & c\alpha^{sr}b\bar{X}_{n}^{sr}(s) \\ \\ c\alpha^{rs}a\bar{X}_{n}^{rs}(s) & R_{an}^{rr} + c\alpha^{r}a\bar{X}_{n}^{rr}(s) \end{bmatrix} \begin{bmatrix} \bar{I}_{a}^{s}(s) \\ \\ \bar{I}_{a}^{r}(s) \end{bmatrix}
$$

...5.33

From which

$$
\bar{I}_{a}^{s}(s) = \frac{1}{|Z(s)|} \left[\left(R_{an}^{rr} + c\alpha^{r}a\bar{X}_{n}^{rr}(s) \right) \bar{v}_{an}^{s}(s) - \bar{v}_{an}^{r}(s)c\alpha^{sr}b\bar{X}_{n}^{sr}(s) \right]
$$

...5.34

$$
\bar{I}_{a}^{r}(s) = \frac{1}{|Z(s)|} \left[\left(R_{an}^{ss} + c\alpha^{s}b\bar{X}_{n}^{ss}(s) \right) \bar{v}_{an}^{r}(s) - \bar{v}_{an}^{s}(s)c\alpha^{sr}a\bar{X}_{n}^{rs}(s) \right]
$$

...5.35

where

$$
|Z(s)| = \left[\left(R_{an}^{ss} + c\alpha^{s}b\bar{X}_{n}^{ss}(s) \right) \left(R_{an}^{rr} + c\alpha^{r}a\bar{X}_{n}^{rr}(s) \right) \right.
$$
$$
\left. - c^{2}\alpha^{sr}\alpha^{sr}ba\,\bar{X}_{n}^{sr}(s)\,\bar{X}_{n}^{rs}(s) \right]
$$

...5.36

Let

$$\bar{X}_n^{ss}(r,s) = \frac{\left[K_n'(\kappa a)\bar{I}_n(\kappa r) - I_n'(\kappa a)K_n(\kappa r)\right]}{\left[K_n'(\kappa a)I_n'(\kappa b) - I_n'(\kappa a)K_n'(\kappa b)\right]} \qquad \ldots 5.37$$

$$X_n^{ss}(r,\omega_s) = \frac{\left[J_n'(m_s a)Y_n(m_s r) - Y_n'(m_s a)J_n'(m_s r)\right]}{\left[J_n'(m_s a)Y_n'(m_s b) - Y_n'(m_s a)J_n'(m_s b)\right]} \qquad \ldots 5.38$$

$$\bar{X}_n^{rr}(r,s) = \frac{\left[K_n'(\kappa b)I_n(\kappa r) - I_n'(\kappa b)K_n(\kappa r)\right]}{\left[K_n'(\kappa a)I_n'(\kappa b) - I_n'(\kappa a)K_n'(\kappa b)\right]} \qquad \ldots 5.39$$

$$X_n^{rr}(r,\omega_r) = \frac{\left[J_n'(m_r b)Y_n(m_r r) - Y_n'(m_r b)J_n(m_r r)\right]}{\left[J_n'(m_r a)Y_n'(m_r b) - Y_n'(m_r a)J_n'(m_r b)\right]} \qquad \ldots 5.40$$

Using the above equations with (4.13) and (4.15) one can write

$$\bar{A}_{an}^s(r,\theta,s) = \left[\frac{\mu_0 N_n^s}{\kappa} \frac{\bar{X}_n^{ss}(r,s)}{|Z(s)|}\left(R_{an}^{rr} + c\alpha^r a \bar{X}_n^{rr}(s)\right)\bar{v}_{an}^s(s) - \bar{v}_{an}^r(s)c\alpha^{sr}b\bar{X}_n^{sr}(s)\right.$$

$$\left. - \mu_0 N_n^s I^s \frac{(sR_{oa}^s + R_{ia}^s)}{c^2(\kappa^2 + m_s^2)}\left(\frac{\bar{X}_n^{ss}(r,s)}{\kappa} - \frac{X_n^{ss}(r,\omega_s)}{m_s}\right)\right]$$

$$\ldots 5.41$$

$$\bar{A}_{an}^{r}(r,\theta,s) = \left[\frac{\mu_0 N_n^{r} \; \bar{X}_n^{rr}(r,s)}{\kappa} \frac{}{|Z(s)|} \left[\left(R_{an}^{ss} + c\alpha^s b \bar{X}_n^{ss}(s) \right) \bar{V}_{an}^{r}(s) - \bar{V}_{an}^{s}(s) c\alpha^s \bar{a} \bar{X}_n^{rs}(s) \right] \right.$$

$$\left. - \mu_0 N_n^{r} I^{r} \frac{(sR_{oa}^{r} + R_{1a}^{r})}{c^2(\kappa^2 + m_r^2)} \left(\frac{\bar{X}_n^{rr}(r,s)}{\kappa} - \frac{X_n^{rr}(r,\omega_r)}{m_r} \right) \right]$$

...5.42

By a similar procedure $\bar{I}_{bn}^{s}(s)$ and $\bar{I}_{bn}^{r}(s)$ may be obtained to determine $\bar{A}_{bn}^{s}(r,\theta,s)$ and $\bar{A}_{bn}^{r}(r,\theta,s)$.

5.5 <u>Discussion</u>

The equations of this chapter are suitable for the study of the machine performance characteristics at starting. In the last section if $\bar{V}_{an}^{ss}(s)$ is a known function and R_{an}^{ss} and R_{an}^{rr} are known parameters, then in principle the inverse transforms may be evaluated immediately. In general these may not be known explicitly and interconnections of the harmonic windings may be necessary to obtain the final solution in the time domain. However, for a machine if the harmonics are negligible, then the equations are immediately applicable for the space fundamental.

CHAPTER 6

ELECTROMAGNETIC ANGULAR MOMENTUM

FLOW AND MACHINE TORQUE

6.1 Introduction

As in any electromagnetic system, the law of conservation of electro-magnetic angular momentum may be conveniently used to determine the production of torque in a machine. With the permeability of iron being considered infinitely larger than that of air, the field quantities are vanishingly small inside the iron parts of the machine and mementum can only be stored in the air gap. The angular momentum flow balance is then uniquely given by equation (1.17A) such that

$$T_{em} = \int_S \vec{r} \times (M.\vec{n})\, dS - \frac{\partial}{\partial t} \int_{V=V_{rotor}} g_\omega\, dV$$

where V_{rotor} is the volume of the rotor and V is the volume enclosed by the surface of integration. Of course, the field quantities are referred to the same co-ordinate system.

If $V = V_o$, i.e., the surface of integration coincides with that of the rotor,

$$T_{em}^r(t) = \int_{S_{rotor}} \vec{r} \times (M.\vec{n})\, dS \qquad \qquad ...6.1$$

gives the mechanical torque on the rotor. The stator torque is the sum of this torque and the rate of change of angular mementum in the air gap such that

$$T_{em}^S(t) = \int_{S_{rotor}} \vec{r} \times (M.\vec{n})\, dS + \frac{\partial}{\partial t} \int_{V_{air\ gap}} g_\omega\, dV \qquad ...6.2$$

For sinusoidal steady state the time averages of T_{em}^r and T_{em}^s are equal and is given by

$$T_{em} = \int_S \vec{r} \times (M.\vec{n}) dS \Big|_{av} \qquad \text{... 6.3}$$

with S being placed anywhere in the air gap.

6.2 Integration of $\vec{r} \times (M.\vec{n}) ds$.

From Maxwell's stress tensor it follows that

$$M.\vec{n} = \epsilon_o(\vec{n}.\vec{E})\vec{E} + \mu_o(\vec{n}.\vec{H})\vec{H} - \vec{n}(\frac{1}{2} \epsilon_o E^2 + \frac{1}{2} \mu_o H^2)$$

Since $\vec{E}$ is completely in the z-direction, $\vec{n}.\vec{E} = 0$ and the contribution of

$$\int \vec{r} \times \epsilon_o(\vec{n}.\vec{E})\vec{E} \, dS = 0$$

Also since

$$\vec{r} \times \vec{n} = 0, \quad \int \vec{r} \times \vec{n}(\frac{1}{2} \epsilon_o E^2 + \frac{1}{2} \mu_o H^2) dS = 0$$

Hence,

$$\int \vec{r} \times (M.\vec{n}) dS = \int \mu_o \vec{r} \times (\vec{n}.\vec{H})\vec{H} \, dS$$

$$= \int \mu_o r \vec{i}_r \times (\vec{i}_r H_r H_r + \vec{i}_\theta H_r H_\theta) dS$$

$$= \vec{i}_z \mu_o \int_0^{2\pi} r^2 H_r H_\theta d\theta \qquad \text{... 6.4}$$

Furthermore, it is known that

$$H_r = \frac{1}{r} \frac{\partial A}{\partial \theta} \qquad \text{... 6.5}$$

$$H_\theta = - (\frac{\partial A}{\partial r}) \qquad \text{... 6.6}$$

Using equations (4.13) through (4.16) or the corresponding time

functions and performing the differentiations indicated in equations (6.5) and (6.6), one obtains

$$\vec{H}^{ss}_{an} = \vec{i}_r H_{anr} \cos n\theta^s - \vec{i}_\theta H_{an\theta} \sin n\theta^s \qquad \ldots 6.7$$

$$\vec{H}^{ss}_{bn} = \vec{i}_r H^{ss}_{bnr} \cos n(\theta^s - \pi/2) - \vec{i}_\theta H^{ss}_{bn\theta} \sin n(\theta^s - \pi/2) \qquad \ldots 6.8$$

$$\vec{H}^{rr}_{an} = \vec{i}_r H^{rr}_{anr} \cos n(\theta^s - \phi) - \vec{i}_\theta H^{rr}_{an\theta} \sin n(\theta^s - \phi) \qquad \ldots 6.9$$

$$\vec{H}^{rr}_{bn} = \vec{i}_r H^{rr}_{bnr} \cos n(\theta^s - \phi - \pi/2) - \vec{i}_\theta H^{rr}_{bn\theta} \sin n(\theta^s - \phi - \pi/2) \qquad \ldots 6.10$$

In particular cases the $H^{pq}_{\alpha\beta}$ quantities may be determined in s or t domain from the corresponding vector potentials. Knowing the θ dependence of the magnetic fields and proceeding with the integration indicated in equation (6.4) it can be shown that there would be no interaction between the magnetic field components of the same phase. Furthermore, there would be no resultant contributions due to the interaction between the magnetic fields of the same member and there would be no interaction between harmonics of different orders. Hence

$$\int_S \vec{r} \times (M \cdot \vec{n}) \, dS = i_z \mu_o \ell \int_0^{2\pi} r^2 \, H_r H_\theta \, d\theta$$

$$= i_z \mu_o \ell r^2 \int_0^{2\pi} \left[-H^{ss}_{anr} H^{rr}_{an\theta} \cos n\theta^s \sin n(\theta^s - \phi) \right.$$

$$- H^{ss}_{an\theta} H^{rr}_{anr} \sin n\theta^s \cos n(\theta^s - \phi)$$

$$- H^{ss}_{anr} H^{rr}_{bn\theta} \cos n\theta^s \sin n(\theta^s - \phi - \pi/2)$$

$$- H^{ss}_{an\theta} H^{rr}_{bnr} \sin n\theta^s \cos n(\theta^s - \phi - \pi/2)$$

$$- H^{ss}_{bnr} H^{rr}_{an\theta} \cos n(\theta^s - \pi/2) \sin n(\theta^s - \phi)$$

$$- H^{ss}_{bn\theta} H^{rr}_{anr} \sin n(\theta^s - \pi/2) \cos n(\theta^s - \phi)$$

$$- H^{ss}_{bnr} H^{rr}_{bn\theta} \cos n(\theta^s - \pi/2) \sin n(\theta^s - \phi - \pi/2)$$

$$- H^{ss}_{bn\theta} H^{rr}_{bnr} \sin n(\theta^s - \pi/2) \cos n(\theta^s - \phi - \pi/2) \bigg] d\theta$$

Proceeding,

$$\int_S \vec{r} \times (M \cdot \vec{n}) dS = \vec{i}_z \mu_0 \pi \ell r^2 \bigg[(H^{ss}_{anr} H^{rr}_{an\theta} - H^{ss}_{an} H^{rr}_{anr}) \sin n\phi$$

$$+ (H^{ss}_{anr} H^{rr}_{bn\theta} - H^{ss}_{an\theta} H^{rr}_{bnr}) \sin n(\phi + \pi/2)$$

$$+ (H^{ss}_{bnr} H^{rr}_{an\theta} - H^{ss}_{bn\theta} H^{rr}_{anr}) \sin n(\phi - /2)$$

$$+ (H^{ss}_{bnr} H^{ss}_{bn\theta} - H^{ss}_{bn\theta} H^{rr}_{bnr}) \sin n\phi \bigg] \qquad \ldots 6.11$$

6.3 **To find** $\int g_\omega dV$

$$g_\omega = \frac{\vec{r} \times (\vec{E} \times \vec{H})}{c^2} = \frac{\vec{r} \times \vec{S}}{c^2}$$

where $\vec{S}$ is the Poynting Vector

$$\vec{S} = \vec{i}_r S_r + \vec{i}_\theta S_\theta$$

$$\vec{r} \times \vec{i}_r = 0 \quad \text{and} \quad \vec{i}_r \times \vec{i}_\theta = \vec{i}_z$$

Hence

$$G_\omega = \frac{1}{c^2} \int_V \vec{r} \times \vec{S} \, dV = \vec{i}_z \frac{1}{c^2} \int_V r^2 S_\theta \, dr \, d\theta \, dz$$

$$= \vec{i}_z \frac{\ell}{c^2} \int_V r^2 S_\theta \, dr \, d\theta$$

Therefore, the angular momentum stored in the air gap

$$G_\omega = \vec{i}_z \frac{\ell}{c^2} \int_a^b \int_0^{2\pi} r^2 S_\theta \, dr \, d\theta \qquad \ldots 6.12$$

where

$$S_\theta = (E_{an}^{ss} + E_{bn}^{ss} + E_{an}^{rr} + E_{bn}^{rr})(H_{anr}^{ss} + H_{bnr}^{ss} + H_{anr}^{rr} + H_{bnr}^{rr})$$

$$\ldots 6.13$$

It may be noted that it would be convenient to carry out the integration in (6.12) for the particular cases.

6.4 Stator and Rotor Torque Unbalance.

The Rotor Torque

$$T_{em}^r(t) = \int_S \vec{r} \times (M \cdot \vec{n}) \, dS \quad \text{with } r \to a$$

$$= \lim_{r \to a} \mu_0 \pi \ell \, r^2 \left[(H_{anr}^{ss} H_{an\theta}^{rr} - H_{an\theta}^{ss} H_{anr}^{rr}) \sin n\phi \right.$$

$$+ (H_{anr}^{ss} H_{bn\theta}^{rr} - H_{an\theta}^{ss} H_{bnr}^{rr}) \sin n(\phi + \pi/2)$$

$$+ (H_{bnr}^{ss} H_{an\theta}^{rr} - H_{bn\theta}^{ss} H_{anr}^{rr}) \sin n(\phi - \pi/2)$$

$$\left. + (H_{bnr}^{ss} H_{bn\theta}^{rr} - H_{bn\theta}^{ss} H_{bnr}^{rr}) \sin n\phi \right] \qquad \ldots 6.14$$

The Stator Torque

$$T_{em}^s(t) = \lim_{r \to a} \int_S \vec{r} \times (M \cdot \vec{n}) \, dS + \frac{1}{c^2} \frac{\partial}{\partial t} \int_a^b \int_0^{2\pi} r^2 S_\theta \, dr \, d\theta \qquad \ldots 6.15$$

The Torque Unbalance is given by

$$\frac{\partial}{\partial t} G_\omega = \frac{\partial}{\partial t} \frac{\ell}{c^2} \int_a^b \int_0^{2\pi} r^2 S_\theta \, d\theta \, dr$$

$$\ldots 6.16$$

= Rate of change of stored angular momentum

in the air gap.

As stated before, at sinusoidal steady state the contribution of this term is zero.

CHAPTER 7

SPACE HARMONICS UNDER

QUASI-STATIC APPROXIMATIONS

7.1 Introduction

All the equations so far considered are in terms of stationary
co-ordinate systems and the rotor has been taken as yet stationary.
When the rotor is in motion the determination of electric field
intensities to write the expressions for currents requires the trans-
formation of the vector potentials (since all other field vectors
may be derived from the vector potentials) between the stator and
rotor co-ordinate systems, which are in relative motion.

Consider two inertial frames S and S' in relative motion with a
constant linear velocity (relative) v along the x-axis. If ϕ' is
the scalar potential and A' the vector potential at a point in the
reference system S', it may be shown that under
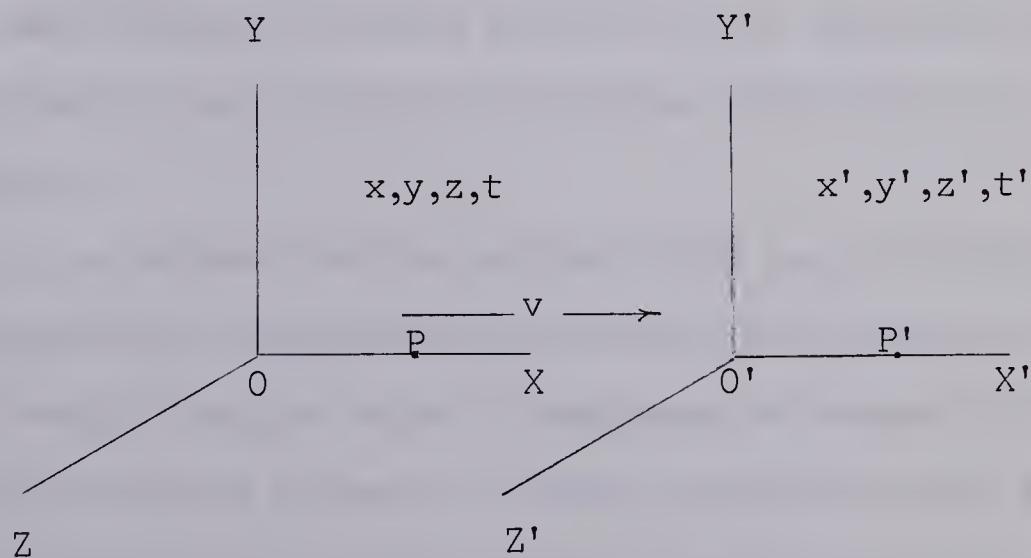
Fig. 7.1 - Inertial Frames

Lorentz Transformation [45] the potentials transform as

$$\Phi = \frac{1}{\sqrt{1 - v^2/c^2}} (\Phi' + v A'_x)$$

$$A_x = \frac{1}{\sqrt{1 - v^2/c^2}} (A'_x + \frac{v}{c^2} \Phi')$$

$$A_y = A'_y, \qquad A_z = A'_z$$

Therefore, for a field problem if $\Phi' = A'_x = A'_y = 0$, the vector potential (with only z-component existing) remains invariant under this transformation such that

$$A_z (x, y, z, t) = A'_z (x', y, z, t')$$

Let $v \ll c$ which will make $x = x' + vt$, $t = t'$ and the length measures in the two systems the same. By applying a transformation of the form

$$\theta = a_1 x \qquad \text{and} \qquad r = a_2 e^{y/a_3}$$

$$(a_1, a_2 \text{ and } a_3 \text{ are arbitrary constants})$$

the S system may be made to coincide with the stator reference system and the S' system to the rotor reference system, which would be in uniform rotation.

Hence, it can be seen that for constant rotor angular velocity ω_m, the vector potentials (z-component only exists in the respective rest system) remain invariant under a transformation between the stator and the rotor co-ordinate systems. To obtain the field vectors from the partial derivatives in a system, stator or rotor, the vector potential may be easily expressed in terms of the co-ordinates of that system. In the stator-system if the rotor position is defined as

$\phi = \omega_m t + \phi_o$, the co-ordinates will transform as

$$\theta^r = \theta^s - \omega_m t - \phi_o \qquad \qquad \ldots 7.1$$

$$r^r = r^s = r \qquad \qquad \ldots 7.2$$

$$z^r = z^s = z \qquad \qquad \ldots 7.3$$

$$t^r = t^s = t \qquad \qquad \ldots 7.4$$

It may be noted that this is similar to the Galilean transformation of the classical mechanics and is valid for the tangential velocity of the field point small in comparison with the velocity of light.

In principle the operational solution of the wave equation can easily accomodate the transformation of the co-ordinates for constant rotor speed. However if the rotor position is given by any arbitrary function of time, it is convenient to use the quasi-static approximations of the vector potentials. The quasi-static approximation, which is in effect the solution of Laplace's equation with the time function written as a factor, has the advantage that frequency independent parameters may be derived (follows from the frequency independence of the co-efficients of the time functions involved in the solution of the field problem) to define the machine equations (the volt ampere equations and the torque equation - collectively known as the equations of motion), which automatically takes care of the motion of the rotor.

The solutions of these equations may determine the machine characteristics (transient and steady state). However, such approximation may not provide an adequate picture of the subtle effects taking place in the electromagnetic system of the machine.

7.2 Quasi-static Approximations of the Vector Potentials

The quasi-static approximations may be obtained by using the low argument approximations (valid at low frequencies only) of the Bessel functions involved in the solutions of the vector potentials. This follows from the fact that for the frequency equal to zero, equation (4.2) or its counterpart in the time domain is reduced to Laplace's equation. Hence in the limit as the frequency tends to zero the solution of either of these equations will be the solution of Laplace's equation with the time function or its transform coming as a factor. This is the quasi-static solution for the vector potential. Now for $z \to 0$, the function[46]

$$I_n(z) = \frac{z^n}{2^n\, n!} \qquad\qquad K_n(z) = 2^{n-1}(n-1)!\, z^{-n}$$

$$J_n(z) = \frac{z^n}{2^n\, n!} \qquad\qquad Y_n(z) = -\frac{2^n(n-1)\, z^{-n}}{\pi}$$

From the above relations it may be shown that

$$\overline{X}_n^{ss}(r,s) = \frac{\left[K_n'(ka)I_n(kr) - I_n'(ka)K_n(kr)\right]}{\left[K_n'(ka)I_n'(kb) - I_n'(ka)K_n'(kb)\right]} = \frac{rk\left[(r/a)^{n-1} + (r/a)^{-n-1}\right]}{n\left[(b/a)^{n-1} - (b/a)^{-n-1}\right]}$$

$$\qquad\qquad \ldots 7.5$$

$$X_n^{ss}(r,\omega) = \frac{\left[J_n'(ma)Y_n(mr) - Y_n'(ma)J_n(mr)\right]}{\left[J_n'(ma)Y_n(mb) - Y_n'(ma)J_n'(mb)\right]} = \frac{rm\left[(r/a)^{n-1} + (r/a)^{-n-1}\right]}{n\left[(b/a)^{n-1} - (b/a)^{-n-1}\right]}$$

$$\qquad\qquad \ldots 7.6$$

$$\overline{X}_n^{rr}(r,s) = \frac{\left[K_n'(\kappa b)I_n(\kappa r) - I_n'(\kappa b)K_n(\kappa r)\right]}{\left[K_n'(\kappa a)I_n'(\kappa b) - I_n'(\kappa a)K_n'(\kappa b)\right]}$$

$$= -\frac{rK}{n}\frac{\left[(r/b)^{n-1} + (r/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]} \qquad \ldots 7.7$$

$$X_n^{rr}(r,\omega) = \frac{\left[J_n'(mb)Y_n(mr) - Y_n'(mb)J_n(mr)\right]}{\left[J_n'(ma)Y_n'(mb) - Y_n'(ma)J_n'(mb)\right]}$$

$$= -\frac{rm}{n}\frac{\left[(r/b)^{n-1} + (r/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]} \qquad \ldots 7.8$$

If these relations are used in equations (4.13) through (4.16), they
are reduced to

$$A_{an}^{ss} = \frac{\mu_0 N_n^s r}{n}\frac{\left[(r/a)^{n-1} + (r/a)^{-n-1}\right]}{\left[(b/a)^{n-1} - (b/a)^{-n-1}\right]}I_{an}^s \sin n\theta^s$$

$$\ldots 7.9$$

$$A_{bn}^{ss} = \frac{\mu_0 N_n^s r}{n}\frac{\left[(r/a)^{n-1} + (r/a)^{-n-1}\right]}{\left[(b/a)^{n-1} - (b/a)^{-n-1}\right]}I_{bn}^s \sin n(\theta^s - \pi/2)$$

$$\ldots 7.10$$

$$A_{an}^{rr} = -\frac{\mu_0 N_n^r r}{n}\frac{\left[(r/b)^{n-1} + (r/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]}I_{an}^r \sin n(\theta^s - \phi)$$

$$\ldots 7.11$$

$$A_{bn}^{rr} = -\frac{\mu_0 N_n^r r}{n} \frac{[(r/b)^{n-1} + (r/b)^{-n-1}]}{[(a/b)^{n-1} - (a/b)^{-n-1}]} I_{bn}^r \sin n\left(\theta^s - \phi - \pi/2\right)$$

$$...7.12$$

The currents I_{an}^s, I_{bn}^s, I_{an}^r and I_{bn}^r will be referred as harmonic currents and may be taken as function of time.

7.3 Volt-Ampere Equations

The air gap electric field intensities are given by

$$E = -\frac{\partial A}{\partial t}$$

(Differentiation in the rest system.)

At the stator surface $r = b$, these intensities are

$$E_{an}^{ss} = -\frac{\partial}{\partial t} \mu_0 N_n^s C_{an}^{ss} I_{an}^s \sin n\theta^s$$

$$E_{bn}^{ss} = -\frac{\partial}{\partial t} \mu_0 N_n^s C_{bn}^{ss} I_{bn}^s \sin n\left(\theta^s - \pi/2\right)$$

$$E_{an}^{sr} = -\frac{\partial}{\partial t} \mu_0 N_n^r C_{an}^{sr} I_{an}^r \sin n\left(\theta^s - \phi\right)$$

$$E_{bn}^{sr} = -\frac{\partial}{\partial t} \mu_0 N_n^r C_{bn}^{sr} I_{bn}^r \sin n\left(\theta^s - \phi - \pi/2\right)$$

And at the rotor surface $r = a$

$$E_{an}^{rs} = -\frac{\partial}{\partial t} \mu_0 N_n^s C_{an}^{rs} I_{an}^s \sin n\theta^s$$

$$E_{bn}^{rs} = -\frac{\partial}{\partial t} \mu_0 N_n^s C_{bn}^{rs} I_{bn}^s \sin n\left(\theta^s - \pi/2\right)$$

$$E_{an}^{rr} = -\frac{\partial}{\partial t} \mu_0 N_n^r C_{an}^{rr} I_{an}^r \sin n\left(\theta^s - \phi\right)$$

$$E_{bn}^{rr} = -\frac{\partial}{\partial t} \mu_0 N_n^r C_{bn}^{rr} I_{bn}^r \sin n\left(\theta^s - \phi - \pi/2\right)$$

where $\quad C_{an}^{ss} = C_{bn}^{ss} = \dfrac{b}{n} \dfrac{[(b/a)^{n-1} + (b/a)^{-n-1}]}{[(b/a)^{n-1} - (b/a)^{-n-1}]}$, $C_{an}^{sr} = C_{bn}^{sr} = -\dfrac{b}{n} \dfrac{2}{[(a/b)^{n-1} - (a/b)^{-n-1}]}$

$$C_{an}^{rr} = C_{bn}^{rr} = -\frac{d}{n}\frac{\left[(a/b)^{n-1} + (a/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]} \quad, \quad C_{an}^{rs} = C_{bn}^{rs} = \frac{a}{n}\frac{2}{\left[(b/a)^{n-1} - (b/a)^{-n-1}\right]}$$

Integrating these intensities as in section (5.3) one can write the volt-ampere equations for the nth harmonic as

$$
\begin{bmatrix} v_{an}^s \\ v_{bn}^s \\ v_{an}^r \\ v_{bn}^r \end{bmatrix}
=
\begin{bmatrix}
R_{an}^s + PL_{an}^s & 0 & PL_{an}^{sr}\cos n\phi & PL_{a.bn}^{sr}\cos n(\phi+\tfrac{\pi}{2}) \\
0 & R_{bn}^s + PL_{bn}^s & PL_{ban}^{sr}\cos n(\phi-\tfrac{\pi}{2}) & PL_{bn}^{sr}\cos n\phi \\
PL_{an}^{sr}\cos n\phi & PL_{bdn}^{sr}\cos n(\phi-\tfrac{\pi}{2}) & R_{an}^r + PL_{an}^r & 0 \\
PL_{abn}^{sr}\cos n(\phi+\tfrac{\pi}{2}) & PL_{bn}^{sr}\cos n\phi & 0 & R_{bn}^r + PL_{bn}^r
\end{bmatrix}
\begin{bmatrix} i_{an}^s \\ i_{bn}^s \\ i_{an}^r \\ i_{bn}^r \end{bmatrix}
$$

$$\ldots 7.13$$

The quantities

$$L_{aan}^{ss} = L_{bbn}^{ss} = L_{an}^s = L_{bn}^s = \frac{\mu_0 \pi \ell b^2 N_n^s N_n^s}{n^2}\frac{\left[(b/a)^{2n}+1\right]}{\left[(b/a)^{2n}-1\right]} \qquad \ldots 7.14$$

$$L_{aan}^{rr} = L_{bbn}^{rr} = L_{an}^r = L_{bn}^r = \frac{\mu_0 \pi \ell a^2 N_n^r N_n^r}{n^2}\frac{\left[1+(a/b)^{2n}\right]}{\left[1-(a/b)^{2n}\right]} \qquad \ldots 7.15$$

$$L_{aan}^{sr} = L_{aain}^{rs} = L_{an}^{sr} \cos n\phi = \frac{\mu_o \pi \ell ab N_n^s N_n^r}{n^2} \frac{2}{\left[(b/a)^n - (b/a)^{-n}\right]} \cos n\phi$$

$$\ldots 7.16$$

$$L_{anbn}^{sr} = L_{bnan}^{rs} = L_{abn}^{sr} = L_{abn}^{sr} \cos n(\phi + \frac{\pi}{2}) = \frac{\mu_o \pi \ell ab N_n^s N_n^r}{n^2} \frac{2}{\left[(b/a)^n - (b/a)^{-n}\right]} \cdot$$

$$\left[\cos n(\phi + \frac{\pi}{2})\right] \qquad \ldots 7.17$$

$$L_{bnan}^{sr} = L_{anbn}^{rs} = L_{ban}^{sr} \cos n(\phi - \frac{\pi}{2}) = \frac{\mu_o \pi \ell ab N_n^s N_n^r}{n^2} \frac{2}{\left[(b/a)^n - (b/a)^{-n}\right]} \cos n(\phi - \frac{\pi}{2})$$

$$\ldots 7.18$$

$$L_{bnbn}^{sr} = L_{bnbn}^{rs} = L_{bn}^{sr} \cos n\phi = \frac{\mu_o \pi \ell ab N_n^s N_n^r}{n^2} \frac{2}{\left[(b/a)^n - (b/a)^{-n}\right]} \cos n\phi$$

$$\ldots 7.19$$

are the nth harmonic inductance parameters (for one coil of the

n coils). The operator p stands for time differentiation. Note

that the inductances expressed as functions of the rotor position ϕ

which is time dependent. If p denotes total differentiation, the

effect of motion on the electrical system is automatically included

in the electrical equations.

The equations (7.13) are a set of nonlinear differential equa-

tions, the solution of which is still a great problem. For any

arbitrary set of conditions a total solution may not be feasible

even if only one harmonic is present. If the rotor position is known

as an explicit function of time, the equations are reduced to linear

differential equations with time varying co-efficients and may be

solved by suitable changes of variables (the so called co-ordinate

transformation of the generalized machine theory).

For transient analysis at constant rotor speed the equations may be referred to a set of co-ordinate systems which are stationary with respect to each other and the method of the Laplace transform may be conveniently used. Transients at constant speed implies that the moment of inertia of the rotor and the attached apparatus is so large that in the limit there is no change of rotor speed during the interval of time considered.

7.4 The Torque Equation

The magnetic field intensities are

$$\vec{H}^{ss}_{aan} = \vec{i}_r N_n^s \frac{[(r/a)^{n-1} + (r/a)^{-n-1}]}{[(b/a)^{n-1} - (b/a)^{-n-1}]} I^s_{an} \cos n\theta^s$$

$$- \vec{i}_\theta N_n^s \frac{[(r/a)^{n-1} - (r/a)^{-n-1}]}{[(b/a)^{n-1} - (b/a)^{-n-1}]} I^s_{an} \sin n\theta^s$$

$$\vec{H}^{ss}_{bbn} = \vec{i}_r N_n^s \frac{[(r/a)^{n-1} + (r/a)^{-n-1}]}{[(b/a)^{n-1} - (b/a)^{-n-1}]} I^s_{bn} \cos n(\theta^s - \pi/2)$$

$$- \vec{i}_\theta N_n^s \frac{[(r/a)^{n-1} - (r/a)^{-n-1}]}{[(b/a)^{n-1} - (b/a)^{-n-1}]} I^s_{bn} \sin n(\theta^s - \pi/2)$$

$$\vec{H}^{rr}_{aan} = -\vec{i}_r N_n^r \frac{[(r/b)^{n-1} + (r/b)^{-n-1}]}{[(a/b)^{n-1} - (a/b)^{-n-1}]} I^r_{an} \cos n(\theta^s - \phi)$$

$$+ \vec{i}_\theta N_n^r \frac{[(r/b)^{n-1} - (r/b)^{-n-1}]}{[(a/b)^{n-1} - (a/b)^{-n-1}]} I^r_{an} \sin n(\theta^s - \phi)$$

$$\vec{H}_{bbn}^{rr} = -\vec{1}_r N_n^r \frac{\left[(r/b)^{n-1} + (r/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]} I_{bn}^r \cos n(\theta^s - \phi - \pi/2)$$

$$+ \vec{1}_\theta N_n^r \frac{\left[(r/b)^{n-1} - (r/b)^{-n-1}\right]}{\left[(a/b)^{n-1} - (a/b)^{-n-1}\right]} I_{bn}^r \sin n(\theta^s - \phi - \pi/2)$$

Put the radial and the tangential components of the magnetic field intensities in equation (6.11) to obtain

$$\int_S \vec{r} \times (M.\vec{n}) \, dS = -L^{sr} \left[I_{an}^s I_{an}^r \sin n\phi \right.$$

$$+ I_{an}^s I_{bn}^r \sin n(\phi + \pi/2)$$

$$+ I_{bn}^s I_{an}^r \sin n(\phi - \pi/2)$$

$$+ \left. I_{bn}^s I_{bn}^r \sin n\phi \right]$$

$$...7.20$$

where,

$$L^{sr} = \frac{2\mu_0 \pi \, l \, a \, b \, N_n^s \, N_n^r}{\left[(b/a)^n - (b/a)^{-n}\right]}$$

If the rate of change of electromagnetic angular momentum storage is neglected, the electromagnetic torque for the nth harmonic is

$$T_{emn}^r(t) = -L^{sr} \left[I_{an}^s I_{an}^r \sin n\phi + I_{an}^s I_{bn}^r \sin n(\phi + \pi/2) \right.$$

$$+ \left. I_{bn}^s I_{an}^r \sin n(\phi - \pi/2) + I_{bn}^s I_{bn}^r \sin n\phi \right]$$

$$...7.21$$

The torque equation of the machine is

$$\sum_{n=1,3,\cdots} T^{r}_{e\,mn} = J_m \frac{d^2\phi}{dt^2} + \alpha_m \frac{d\phi}{dt} + K_m\phi + T_{mech} \qquad \ldots 7.22$$

where,

J_m = moment of inertia of the rotating parts.

α_m = rotational frictional co-efficient

K_m = spring constant

T_{mech} = Output mechanical torque

7.5 Discussion

Harmonic analysis of a two-pole, two-phase cylindrical rotor machine under quasi-static approximations is shown in this chapter and the method may be easily extended to any cylindrical rotor machine. The results may be suitably applied to any particular case by relating the harmonic currents and voltages to the actual winding currents and terminal voltages of the physical device.

It may be noted that White and Woodson[47] have obtained these results as an extension of their generalized machine analysis. Their method includes the solution of Laplace's equation in the air gap, and the determination of the inductances through energy storage. The equations of motion are written by using the Euler-Lagrange equation of the classical mechanics. They have also shown how to write the volt-ampere equations through the concept of rate of charge of flux linkages and deduced the torque equation by using the principle of conservation of energy and virtual displacement.

As may be summarized the method shown here includes approximation of the solution of the vector wave equation to obtain the quasi-static solutions for the vector potentials, from which all other field quantities are derived by simple operations. The volt-ampere equations are written by using the line integrals of the electric field intensities. The torque equation is obtained by using Maxwell's stress tensor and applying the principle of conservation of angular momentum (neglecting the effect of rate of change of electromagnetic angular momentum storage) in an electromagnetic system. The end results are the same as those of White and Woodson.

CHAPTER 8

STEADY STATE ANGULAR MOMENTUM

FLOW IN AN INDUCTION MOTOR

8.1 Introduction

The theory thus far developed will now be used for the detailed
analysis of a two-phase, two-pole wound rotor induction machine to
show the electromagnetic angular momentum flow at the steady state.
The cases of a machine with a conducting sleeve on the rotor iron
and a similar machine without any iron in the stator or rotor will
be introduced to further enhance the concept of the electromagnetic
angular momentum flow. It will be considered that the stator excita-
tion is balanced and the rotor is running at constant angular
velocity $\omega_m (\omega_m = 0$ is a special case). Actually the induction motor
is not a constant-speed machine. However, in most of the practical
machines, the mechanical response (i.e. the rate of acceleration of
the rotor) is much slower than the electrical response; hence, a
constant speed constraint is legitimate for this kind of machines.

Under the assumption of balanced excitation, let the stator
currents be specified as

$$i_a^S (t) = I^S \cos(\omega_s t - \theta_p^S) \qquad \text{...8.1}$$

$$i_b^S (t) = I^S \sin(\omega_s t - \theta_p^S) \qquad \text{...8.2}$$

A self consistent analysis will show that for constant rotor speed

the rotor currents will also be balanced and let these currents be

$$i_a^r(t) = I^r \cos(\omega_r t - \theta_p^r) \qquad \ldots 8.3$$

$$i_b^r(t) = I^r \sin(\omega_r t - \theta_p^r) \qquad \ldots 8.4$$

8.2 Inverse Transforms of the Stator Vector Potentials

Since only the steady state field components are required to study the electromagnetic angular momentum flow at the steady state, the evaluation of the inverse transforms in this section is limited to obtain the steady state stator vector potentials. In the inversion process these functions are contributed by the poles of the transforms of the driving functions and may be determined as follows.

If the initial conditions are zero, then for n = 1, equations (4.13) and (4.14) are reduced to

$$\bar{A}_{d1}^s(r,\theta^s,s) = \mu_0 N_1^s \frac{c}{s} \bar{X}_1^{ss}(r,s) \bar{I}_a^s(s) \sin \theta^s \qquad \ldots 8.5$$

and

$$\bar{A}_{b1}^s(r,\theta^s,s) = \mu_0 N_1^s \frac{c}{s} \bar{X}_1^{ss}(r,s) \bar{I}_b^s(s) \sin(\theta^s - \pi/2) \qquad \ldots 8.6$$

where, $\bar{X}_1^{ss}(r,s)$ is given by

$$\bar{X}_1^{ss}(r,s) = \frac{\left[K_1'(sa/c) I_1(sr/c) - I_1'(sa/c) K_1(sr/c)\right]}{\left[K_1'(sa/c) I_1'(sb/c) - I_1'(sa/c) K_1'(sb/c)\right]} \qquad \ldots 8.7$$

The Laplace transforms of the currents in equations (8.1) and (8.2)

are

$$\bar{I}_a^S(s) = \int_0^\infty I^S e^{-st} \cos(\omega_s t - \theta_P^S) dt$$

$$= \frac{(s\cos\theta_P^S + \omega_s \sin\theta_P^S)}{(s^2 + \omega_s^2)} \qquad \ldots 8.8$$

and

$$\bar{I}_b^S(s) = \int_0^\infty I^S e^{-st} \sin(\omega_s t - \theta_P^S) dt$$

$$= \frac{(\omega_s \cos\theta_P^S - s\sin\theta_P^S)}{(s^2 + \omega_s^2)} \qquad \ldots 8.9$$

Substitution of these into equations (8.5) and (8.6) yield

$$\bar{A}_{a1}^S(r,\theta^S,s) = \mu_0 N_1^S I^S \frac{c}{s} \bar{X}_1^{SS}(r,s) \sin\theta^S \frac{(s\cos\theta_P^S + \omega_s \sin\theta_P^S)}{(s^2 + \omega_s^2)}$$

$$\ldots 8.10$$

$$\bar{A}_{b1}^S(r,\theta^S,s) = \mu_0 N_1^S I^S \frac{c}{s} \bar{X}_1^{SS}(r,s) \sin(\theta^S - \pi/2) \frac{(\omega_s \cos\theta_P^S - s\sin\theta_P^S)}{(s^2 + \omega_s^2)}$$

$$\ldots 8.11$$

Using the inversion theorem

$$A_{a1}^S(r,\theta^S,t) = \frac{\mu_0 N_1^S c I^S \sin\theta^S}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{\bar{X}_1^{SS}(r,s)}{s} \frac{(s\cos\theta_P^S + \omega_s \sin\theta_P^S)}{(s^2 + \omega_s^2)} e^{st} ds$$

$$\ldots 8.12$$

$$A_{b1}^S(r,\theta^S,t) = -\frac{\mu_0 N_1^S c I^S \cos\theta^S}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{\bar{X}_1^{SS}(r,s)}{s} \frac{(\omega_s \cos\theta_P^S - s\sin\theta_P^S)}{(s^2 + \omega_s^2)} e^{st} ds$$

$$\ldots 8.13$$

In the above inversion integrals, the poles at $s = \pm j\omega_s$ will determine the steady state functions. To apply Cauchy's Residue Theorem for the evaluation of these functions, close the contour such that it encloses the poles of the integrands at $s = \pm j\omega_s$ but does not pass through any other pole. Take the residues associated with these poles.

Let

$$X_1^{ss}(r,\omega) = \frac{[J_1'(\omega_s a/c)\, Y_1(\omega_s r/c) - Y_1'(\omega_s a/c)\, J_1(\omega_s r/c)]}{[J_1'(\omega_s a/c)\, Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\, J_1'(\omega_s b/c)]} \quad \ldots 8.14$$

Using the properties of Bessel functions, it may be shown that

$$\bar{X}_1^{ss}(r, j\omega_s) = j\, X_1^{ss}(r, \omega_s)$$

and

$$\bar{X}_1^{ss}(r, -j\omega_s) = -j\, X_1^{ss}(r, \omega_s)$$

For $A_{a1}^{S}(r, \theta^S, t)$, the residue evaluated at $s = j\omega_s$

$$= -\frac{\mu_0 c\, N_1^S I^S X_1^{ss}(r, \omega_s)}{(2\pi j)\,(2\omega_s)} \sin\theta^S (\cos\theta_P^S - j\sin\theta_P^S)\, e^{j\omega_s t}$$

and that at $s = -j\omega_s$

$$= -\frac{\mu_0 c\, N_1^S I^S X_1^{ss}(r, \omega_s)}{(2\pi j)\,(2\omega_s)} \sin\theta^S (\cos\theta_P^S + j\sin\theta_P^S)\, e^{-j\omega_s t}$$

Summing up the above residues and multiplying by $2\pi j$,

$$A_{a1}^{S}(r, \theta^S, t) = \mu_0 N_1^S I^S(\tfrac{c}{\omega_s}) X_1^{ss}(r, \omega_s) \sin\theta^S \cos(\omega_s t - \theta_P^S) \quad \ldots 8.15$$

Similarly,

$$A_{b1}^S(r,\theta,t) = -\mu_o N_1^S I^S\left(\frac{c}{\omega_s}\right) X_1^{SS}(r,\omega_s) \cos\theta^S \sin(\omega_s t - \theta_P^S) \quad \ldots 8.16$$

Adding equations (8.15) and (8.16) one obtains the resultant steady state air gap vector potential due to the stator currents

$$A_{t1}^S(r,\theta,t) = \mu_o N_1^S I^S\left(\frac{c}{\omega_s}\right) \frac{[J_1'(\omega_s a/c) Y_1(\omega_s r/c) - Y_1'(\omega_s a/c) J_1(\omega_s r/c)]}{[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c)]} \sin(\theta^S - \omega_s t - \theta_P^S)$$

$$\ldots 8.17$$

8.3 Inverse Transforms of the Rotor Vector Potentials

Setting the initial conditions equal to zero and n = 1, equations (4.15) and (4.16) may be simplified as

$$\bar{A}_{a1}^r(r,\theta^r,s) = \mu_o N_1^r\left(\frac{c}{s}\right) \bar{X}_1^{rr}(r,s) \bar{I}_a^r(s) \sin\theta^r \quad \ldots 8.18$$

$$\bar{A}_{b1}^r(r,\theta^r,s) = -\mu_o N_1^r\left(\frac{c}{s}\right) \bar{X}_1^{rr}(r,s) \bar{I}_b^r(s) \cos\theta^r \quad \ldots 8.19$$

where,

$$\bar{X}_1^{rr}(r,s) = \frac{[K_1'(sb/c) I_1(sr/c) - I_1'(sb/c) K_1(sa/c)]}{[K_1'(sa/c) I_1'(sb/c) - I_1'(sa/c) K_1'(sb/c)]}$$

$$\ldots 8.20$$

Following exactly the same procedure as in the previous section, it may be shown that for the steady state

$$A_{a1}^r(r,\theta^r,t) = \mu_o N_1^r I^r\left(\frac{c}{\omega_r}\right) X_1^{rr}(r,\omega_r) \sin\theta^r \cos(\omega t - \theta_P^r) \quad \ldots 8.21$$

and

$$A^r_{b1}(r,\theta,t) = -\mu_0 N^r_1 I^r(\tfrac{c}{\omega_r}) X^{rr}_1(r,\omega_r) \cos \theta^r \sin(\omega_r t - \theta^r_p)$$

...8.22

where

$$X^{rr}_1(r,\omega_r) = \frac{[J'_1(\omega_r b/c) Y_1(\omega_r r/c) - Y'_1(\omega_r b/c) J_1(\omega_r r/c)]}{[J'_1(\omega_r a/c) Y'_1(\omega_r b/c) - Y'_1(\omega_r a/c) J'_1(\omega_r b/c)]}$$

...8.23

The resultant steady state air gap vector potential due to the rotor currents is

$$A^r_{t1}(r,\theta,t) = \mu_0 N^r_1 I^r(\tfrac{c}{\omega_r}) \frac{[J'_1(\omega_r b/c) Y_1(\omega_r r/c) - Y'_1(\omega_r b/c) J_1(\omega_r r/c)]}{[J'_1(\omega_r a/c) Y'_1(\omega_r b/c) - Y'_1(\omega_r a/c) J'_1(\omega_r b/c)]} \sin(\theta^r - \omega_r t - \theta^r_p)$$

...8.24

(It may be shown that for any of the currents specified in section (8.1), if there were an nth space harmonic, the time domain solution of the corresponding vector potential will be of the form shown in this and the previous sections except for a change in N and the order of the Bessel functions.)

8.4 The Currents

The magnitudes and phase angles of the currents will now be calculated. Indeed, it will follow that the assumption of the forms of the current expressions in section (8.1) is self consistent.

(A) Rotor Currents

Using the transformations of equations (7.1) through (7.4), with respect to the rotor, equation (8.17) may be written as

$$A_{t1}^{rs}(r,\theta^r,t) = \mu_o N_1^{S-S_o} \frac{\left[J_1'(\omega_r a/c)Y_1(\omega_r r/c) - Y_1'(\omega_r a/c)J_1(\omega_r r/c)\right]}{\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c)J_1'(\omega_r b/c)\right]}$$

$$\left[\sin(\theta^r - \omega_r t + \theta_p^S + \phi_o)\right] \quad\quad ...8.25$$

where
$$\omega_\gamma = \omega_S - \omega_m$$

From (8.24) and (8.25) the electric field intensities at the rotor surface are

$$E_{t1}^{rr} = \mu_o c \, N_1^r I^r \frac{\left[J_1'(\omega_r b/c)Y_1'(\omega_r a/c) - Y_1'(\omega_r b/c)J_1(\omega_r a/c)\right]}{\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c)J_1'(\omega_r b/c)\right]}$$

$$\left[\cos(\theta^r - \omega_r t + \theta_p^r)\right] \quad\quad ...8.26$$

$$E_{t1}^{rs} = \mu_o c \, N_1^S I^S \frac{\left[J_1'(\omega_r a/c)Y_1(\omega_r a/c) - Y_1'(\omega_r a/c)J_1(\omega_r a/c)\right]}{\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c)J_1'(\omega_r b/c)\right]}$$

$$\left[\cos(\theta^r - \omega_r t + \theta_p^S + \phi_o)\right] \quad\quad ...8.27$$

Their sum gives the total electric field intensity at the rotor surface such that,

$$E_{t1}^r = E_{t1}^{rr} + E_{t1}^{rs}$$

since the rotor is short circuited this is the only field intensity which is to be equated to the rotor resistive drop per unit length along the z-direction

$$E_{t1}^r = R_{\ell 1}^r I^r \cos(\omega_r t - \theta_p^r) \quad\quad ...8.28$$

Integrating this equation as in section (5.3) one can show that

for the rotor phase (a)

$$I^r R^r \cos(\omega_r t - \theta_p^r) - \omega_r L^{rr} I^r \sin(\omega_r t - \theta_p^r)$$

$$= \omega_r L^{rs} I^s \sin(\omega_r t - \theta_p^s - \phi_o)$$

...8.29

where

R^r = Total rotor resistance per phase

$$L^{rr} = \mu_o \pi \ell a N_1^r {}^r N_1^r (\frac{c}{\omega_r}) \frac{\left[J_1'(\omega_r b/c) Y_1(\omega_r a/c) - Y_1'(\omega_r b/c) I_1(\omega_r a/c) \right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c) \right]}$$

...8.30

$$L^{rs} = \mu_o \pi \ell a N_1^r N_1^s (\frac{c}{\omega_r}) \frac{\left[J_1'(\omega_r a/c) Y_1(\omega_r a/c - Y_1'(\omega_r a/c) J_1(\omega_r a/c) \right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c) \right]}$$

...8.31

The L quantities are inductances.

From (8.29)

$$I^r R^r \cos(\omega_r t - \theta_p^r) + j\omega_r L^{rr} I^r \cos(\omega_r t - \theta_p^s - \phi_o)$$

$$= \omega_r L^{rs} I^s \sin(\omega_r t - \theta_p^s - \phi_o)$$

And from this,

$$I^r = \frac{\omega_r L^{rs} I^s}{((R^r)^2 + (\omega_r L^{rr})^2)^{1/2}}$$

...8.32

$$\theta_P^r = \theta_P^s + \phi_0 + \psi_r + \pi/2 \qquad \qquad \dots 8.33$$

where, $\qquad \qquad \Upsilon_r = \tan^{-1}\dfrac{\omega_r L^{rr}}{R_r} = $ rotor impedance angle $\qquad \dots 8.34$

Similar expressions will follow for I^r and θ_P^r if the rotor phase

(b) is considered giving self consistent results.

(B) Stator Currents

    With respect to the stator, the rotor vector potential is

$$A_{t1}^{sr}(\Upsilon,\theta^s,t) = \mu_0 N_1^r I_1^r \left(\frac{c}{\omega_s}\right) \frac{\left[J_1'(\omega_s b/c) Y_1(\omega_s r/c) - Y_1'(\omega_s b/c) J_1(\omega_s r/c) \right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]}.$$

$$\left[\sin(\theta^s - \omega_s t + \theta_P^r - \phi_0) \right] \qquad \dots 8.35$$

From (8.17) and (8.35), the electric field intensities at the stator

surface r = b, are

$$E_{t1}^{ss} = \mu_0 c \, N_1^s I^s \frac{\left[J_1'(\omega_s a/c) Y_1(\omega_s b/c) - Y_1'(\omega_s a/c) J_1(\omega_s b/c) \right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]}.$$

$$\left[\cos(\theta^s - \omega_s t + \theta_P^s) \right] \qquad \dots 8.36$$

$$E_{t1}^{sr} = \mu_0 c \, N_1^r I^r \frac{\left[J_1'(\omega_s b/c) Y_1(\omega_s b/c) - Y_1'(\omega_s b/c) J_1(\omega_s b/c) \right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]}.$$

$$\left[\cos(\theta^s - \omega_s t - \phi_0) \right] \qquad \dots 8.37$$

and the total

$$E_{t1}^s = E_{t1}^{ss} + E_{t1}^{sr}$$

Let the applied voltages be

$$v_a^s(t) = V^s \cos \omega_s t \qquad \ldots 8.38$$

$$v_b^s(t) = V^s \sin \omega_s t \qquad \ldots 8.39$$

Integrating the field intensities for the stator phase (a), it may be written that

$$R^s I^s \cos(\omega_s t - \theta^s) = \int_{\ell_p} E_{t1}^s d\ell_p + V^s \cos \omega_s t$$

$$R^s I^s \cos(\omega_s t - \theta^s) - \omega_s L^{ss} I^s \left(\sin(\omega_s t - \theta_p^s)\right)$$

$$= \omega_s L^{sr} I^r \sin(\omega_s t - \theta_p^r + \phi_0) + V^s \cos \omega_s t \qquad \ldots 8.40$$

where,

R^s = Total stator resistance per phase

$$L^{ss} = \mu_0 \pi \ell b \, N_1^s N_1^s (\frac{c}{\omega_s}) \frac{\left[J_1'(\omega_s a/c) Y_1(\omega_s b/c) - Y_1'(\omega_s a/c) J_1(\omega_s b/c) \right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]}$$

$$\ldots 8.30A$$

$$L^{sr} = \mu_0 \pi \ell b \, N_1^s N_1^r (\frac{c}{\omega_s}) \frac{\left[J_1'(\omega_s b/c) Y_1(\omega_s b/c) - Y_1'(\omega_s b/c) J_1(\omega_s b/c) \right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]}$$

$$\ldots 8.31A$$

Again the L quantities are inductances.

Equation (8.40) may be solved as follows:

$$(R^s + j\omega_s L^{ss}) I^s \cos(\omega_s t - \theta_P^s)$$

$$= \omega_s L^{sr} I^r \sin(\omega_s t - \theta_P^r + \phi_0) + V^s \cos\omega_s t$$

Let

$$R^s + j\omega_s L^{ss} = |Z_s| e^{j\psi_s}$$

where

$$|Z_s| = \sqrt{(R^s)^2 + (\omega_s L^{ss})^2} \quad \text{and} \quad \psi_s = \tan^{-1} \frac{\omega_s L^{ss}}{R^s}$$

Then with exponential notations

$$\left[|Z_s| e^{j\psi_s} I^s e^{-j\theta_P^s} \right] e^{j\omega_s t} = \left[-\omega_s L^{sr} I^r e^{j(\phi_0 - \theta_P^r + \pi/2)} + V^s \right] e^{j\omega_s t}$$

From which

$$\left[|Z_s| e^{j\psi_s} I^s e^{-j\theta_P^s} \right] = \left[-\omega_s L^{sr} I^r e^{j(\phi_0 - \theta_P^r + \pi/2)} + V^s \right]$$

So that

$$I^s e^{-j\theta_P^s} = \frac{1}{|Z_s|} \left[-\omega_s L^{sr} I^r e^{j(\phi_0 - \theta_P^r + \pi/2)} + V^s \right] e^{-j\psi_s}$$

Calculation of I^s

Since $\left| e^{-j\psi_s} \right| = 1$, and

$$\left[V^s - \omega_s L^{sr} I^r e^{j(\phi_0 - \theta_P^r + \phi_0)} \right]$$

$$= \left[V^s - \omega_s L^{sr} I^r \cos(\phi_0 - \theta_P^r + \pi/2) - j\omega_s L^{sr} I^r \sin(\phi_0 - \theta_P^r + \pi/2) \right]$$

Hence

$$I^s = \frac{1}{|Z_s|} \sqrt{\left[(V^s)^2 + (\omega_s L^{sr} I^r)^2 - 2 V^s \omega_s L^{sr} I^r \cos(\phi_0 - \theta_P^r + \pi/2) \right]}$$

...8.41

Calculation of θ_P^S

$$I^S e^{-j\theta_P^S} = \frac{1}{|Z_s|}\left[V^S e^{-jY_s} - \omega_s L^{sr} I^r e^{j(\phi_o - \theta_P^r + \pi/2 - Y_s)}\right]$$

But,

$$V^S e^{-jY_s} = V^S \cos Y_s - j V^S \sin Y_s \ .$$

and,

$$-\left[\omega_s L^{sr} I^r e^{j(\phi_o - \theta_P^r + \pi/2 - Y_s)}\right] = \left[-\omega_s L^{sr} I^r \cos(Y_s - \phi_o + \theta_P^r - \pi/2)\right.$$

$$\left. + j \omega_s L^{sr} I^r \sin(Y_s - \phi_o + \theta_P^r - \pi/2)\right]$$

It follows that

$$\theta_P^S = \tan^{-1}\frac{\left[V^S \sin Y_s - \omega_s L^{sr} I^r \sin(Y_s - \phi_o + \theta_P^r - \pi/2)\right]}{\left[V^S \cos Y_s - \omega_s L^{sr} I^r \cos(Y_s - \phi_o + \theta_P^r - \pi/2)\right]}$$

$$...8.42$$

Similarly

$$i_b^S(t) = I^S \sin(\omega_s t - \theta_P^S)$$

as assumed.

8.5 Rotor Current Density of the Sleeve Rotor Machine

The current density may be calculated as follows:

In equation (8.25) let

$$I^S N_1^S = I_1^S \qquad \text{and} \qquad \theta^r + \theta_P^S + \phi_o = \theta_r$$

So that with exponential notation (the imaginary part gives the solution)

$$A_{t1}^{rs}(r,\theta_r,t) = \mu_o I_1^S\left(\frac{c}{\omega r}\right)\frac{\left[J_1'(\omega_r a/c) Y_1(\omega_r r/c) - Y_1'(\omega_r a/c) J_1(\omega_r r/c)\right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1(\omega_r b/c)\right]} e^{-j(\theta_r - \omega_r t)}$$

$$...8.43$$

By analogy with section (8.2), it follows that this vector
potential may be produced by two equivalent current sheets placed
in space and time quadrature, the space being coincident with the
stator but moving at an angular velocity ω_m with respect to the
stator co-ordinate system. In this space the resultant current
density distribution producing $A_{t1}^{rs}(r,\theta.,t)$ is

$$I_1^s(\theta_r,t) = I_1^s e^{j(\theta r - \omega_r t)} \qquad \ldots 8.44$$

Since the rotor is stationary with respect to co-ordinate system
(r,θ_r,t), the rotor current density will have the frequency ω_r, but
differ in magnitude and phase from the current density in equation
(8.44). Let the resultant rotor current density be

$$I_1^r(\theta_r,t) \doteq I_1^r e^{j(\theta r - \omega_r t + \theta_P^r)} \qquad \ldots 8.45$$

The vector potential due to this current density is

$$A_{t1}^r(r,\theta_r,t) = \mu_0 I_1^r(\tfrac{c}{\omega_r}) \frac{\left[J_1'(\omega_r b/c) Y_1(\omega_r r/c) - Y_1'(\omega_r b/c) J_1(\omega_r r/c)\right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)\right]} e^{j(\theta_r - \omega_r t + \theta_P^r)} \qquad \ldots 8.46$$

To evaluate I_1^r and θ_P^r, equate the electric field intensities
on the rotor surface caused by the stator and the rotor currents
to the ohmic voltage drop per unit length along the rotor such that

$$R_r I_1^r e^{j(\theta_r - \omega_r t + \theta_P^r)} = j\mu_0 c I_1^r \frac{\left[J_1'(\omega_r b/c) Y_1(\omega_r a/c) - Y_1'(\omega_r b/c) J_1(\omega_r a/c)\right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)\right]} e^{j(\theta_r - \omega_r t + \theta_P^r)}$$

$$+ j\mu_0 c I_1^s \frac{\left[J_1'(\omega_r a/c) Y_1(\omega_r a/c) - Y_1'(\omega_r a/c) J_1(\omega_r a/c)\right]}{\left[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)\right]} e^{j(\theta_r - \omega_r t)} \qquad \ldots 8.47$$

where R_r = rotor surface resistivity.

Equating the real and imaginary parts of (8.47), I_1^r and θ_p^r may easily be calculated. Also referring the vector potentials to the stationary stator co-ordinate system, the current density I_1^S and its phase angle may be calculated in terms of terminal voltages. Since the form assumed for the rotor current density in (8.45) leads to self consistent result, the assumption is justified.

8.6 Angular Momentum Flow

Using equations (8.25) and (8.24) one can write with respect to rotor co-ordinate system

$$\vec{H}_{t1}^{rs} = \vec{i}_r N_1^S I^S(\tfrac{c}{\omega_r}) \frac{1}{r} \frac{[J_1'(\omega_r a/c) Y_1(\omega_r r/c) - Y_1'(\omega_r a/c) J_1(\omega_r r/c)]}{[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)]} \cos(\theta^r - \omega_r t + \theta_p^s + \phi_0)$$

$$- \vec{i}_\theta N_1^S I^S \frac{[J_1'(\omega_r a/c) Y_1'(\omega_r r/c) - Y_1'(\omega_r a/c) J_1'(\omega_r r/c)]}{[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)]} \sin(\theta^r - \omega_r t + \theta_p^s + \phi_0)$$

$$\ldots 8.48$$

$$\vec{H}_{t1}^{rr} = \vec{i}_r N_1^r I^r(\tfrac{c}{\omega_r}) \frac{1}{r} \frac{[J_1'(\omega_r b/c) Y_1(\omega_r r/c) - Y_1'(\omega_r b/c) J_1(\omega_r r/c)]}{[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)]} \cos(\theta^r - \omega_r t + \theta_p^r)$$

$$- \vec{i}_\theta N_1^r I^r \frac{[J_1'(\omega_r b/c) Y_1'(\omega_r r/c) - Y_1'(\omega_r b/c) J_1'(\omega_r r/c)]}{[J_1'(\omega_r a/c) Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c) J_1'(\omega_r b/c)]} \sin(\theta^r - \omega_r t + \theta_p^r) \ldots 8.49$$

The time average of $\int_S \vec{r} \times (M.\vec{n}) dS$ will be evaluated with S being a cylindrical surface of radius r, $a \leqslant r \leqslant b$

$$\left. \int_S \vec{r} \times (M.\vec{n}) dS \right|_{av} = \mu_0 \ell \left. \int_0^{2\pi} r^2 H_\theta H_r d\theta \right|_{av}$$

$$= \mu_0 \ell \left. \int_0^{2\pi} r^2 (H_{t1\theta}^{rs} H_{t1r}^{rr} + H_{t1r}^{rs} H_{t1\theta}^{rr}) d\theta \right|_{av}$$

However,

$$\int_0^{2\pi} \sin(\theta^r - \omega_r t + \theta_p^S + \phi_0)\cos(\theta^r - \omega_r t + \theta_p^r)\,d\theta\bigg|_{av} = \pi \sin(\theta_p^S + \phi_0 - \theta_p^r)$$

$$\int_0^{2\pi} \sin(\theta^r - \omega_r t + \theta_p^r)\cos(\theta^r - \omega_r t + \theta_p^S + \phi)\,d\theta\bigg|_{av} = -\pi \sin(\theta_p^S + \phi_0 - \theta_p^r)$$

and

$$\left[J_1'(\omega_r b/c)Y_1'(\omega_r r/c) - Y_1'(\omega_r b/c)J_1'(\omega_r r/c)\right]\left[J_1'(\omega_r a/c)Y_1(\omega_r r/c) - Y_1'(\omega_r a/c)J_1(\omega_r r/c)\right]$$

$$- \left[J_1'(\omega_r b/c)Y_1(\omega_r r/c) - Y_1'(\omega_r b/c)J_1(\omega_r r/c)\right]\left[J_1'(\omega_r a/c)Y_1'(\omega_r r/c) - Y_1'(\omega_r a/c)J_1'(\omega_r r/c)\right]$$

$$= \left[J_1(\omega_r r/c)Y_1'(\omega_r r/c) - Y_1(\omega_r r/c)J_1'(\omega_r r/c)\right]\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c)J_1'(\omega_r b/c)\right]$$

$$= \frac{a}{r}\left[J_1(\omega_r a/c)Y_1'(\omega_r a/c) - Y_1(\omega_r a/c)J_1'(\omega_r a/c)\right]\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - Y_1'(\omega_r a/c)J_1'(\omega_r b/c)\right]$$

(The above simplication has used the Wronskian relation[48]

$$J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z) = \frac{2}{\pi z}, \text{ all } \nu, z \neq 0)$$

Hence

$$\int_S \vec{r} \times (M \cdot \vec{n})\,dS\bigg|_{av} = \mu_0 \pi\, a l\, N_1^S N_1^r I^S I^r \left(\frac{c}{\omega_r}\right)\sin(\theta_p^S + \phi_0 - \theta_p^r) \cdot$$

$$\left[\frac{\left[J_1(\omega_r a/c)Y_1'(\omega_r a/c) - J_1'(\omega_r a/c)Y_1(\omega_r a/c)\right]}{\left[J_1'(\omega_r a/c)Y_1'(\omega_r b/c) - J_1'(\omega_r b/c)Y_1'(\omega_r a/c)\right]}\right]$$

$$= L^{rs}I^S I^r \sin(\theta_p^r - \theta_p^S - \phi_0) \qquad \ldots 8.50$$

where L^{rs} is given by (8.31).

Equation (8.50) gives the average rate at which electromagnetic angular momentum is crossing the air gap of the machine and is absorbed on the rotor surface. Since the time average of

$$\frac{\partial}{\partial t} \mu_o \epsilon_o \int_V \vec{r} \times (\vec{E} \times \vec{H}) dV$$

is zero, the average mechanical torque (known as the electromagnetic torque) on the rotor is

$$T^r_{em} = \int_S \vec{r} \times (M.\vec{n}) dS \bigg|_{av} = L^{rs} I^s I^r \sin(\theta^r_p - \theta^s_p - \phi_o) \quad ...8.51$$

Expressing the field components in terms of the stator co-ordinate system, it may be readily shown that the magnitude of the reaction torque on the stator is

$$T^s_{em} = \int_S \vec{r} \times (M.\vec{n}) dS \bigg|_{av} = L^{sr} I^s I^r \sin(\theta^r_p - \theta^s_p - \phi_o) \quad ...8.52$$

(the surface S being stationary with respect to the stator)

But $\qquad\qquad L^{sr} = L^{rs}$

Hence $\qquad\qquad T^r_{em} = T^s_{em}$

showing that the rate at which the electromagnetic angular momentum is crossing the air gap is invariant under a transformation of the co-ordinates.

8.7 Power Flow

The average power flow across the air gap is given by

$$P_{av} = - \int_S (\vec{E} \times \vec{H}).\vec{n} \, dS \bigg|_{av} = - \ell r \int_0^{2\pi} S_r d\theta \bigg|_{av} \qquad ...8.53$$

with respect to rotor co-ordinate system

$$\vec{S} = \vec{E} \times \vec{H} = \vec{1}_r S_r + \vec{1}_\theta S_\theta$$

$$\begin{aligned} = -\vec{1}_r \left[E^{rs}_{t1} H^{rs}_{t1\theta} + E^{rs}_{t1} H^{rr}_{t1\theta} + E^{rr}_{t1} H^{rs}_{t1\theta} + E^{rr}_{t1} H^{rr}_{t1\theta} \right] \\ + \vec{1}_\theta \left[E^{rs}_{t1} H^{rs}_{t1r} + E^{rs}_{t1} H^{rr}_{t1r} + E^{rr}_{t1} H^{rs}_{t1r} + E^{rr}_{t1} H^{rr}_{t1r} \right] \end{aligned} \qquad ...8.54$$

where,

$$E_{t1}^{rr} = \mu_0 c N_1^r I^r \frac{\left[J_1'(\omega r b/c) Y_1(\omega r r/c) - Y_1'(\omega r b/c) J_1(\omega r r/c) \right]}{\left[J_1'(\omega r a/c) Y_1'(\omega r b/c) - Y_1'(\omega r a/c) J_1'(\omega r b/c) \right]} \cos(\theta^r - \omega_r t + \theta_p^r)$$

$$\cdots 8.55$$

$$E_{t1}^{rs} = \mu_0 c N_1^s I^r \frac{\left[J_1'(\omega r a/c) Y_1(\omega r r/c) - Y_1'(\omega r a/c) J_1(\omega r r/c) \right]}{\left[J_1'(\omega r a/c) Y_1'(\omega r b/c) - Y_1'(\omega r a/c) J_1'(\omega r b/c) \right]} \cos(\theta^r - \omega_r t + \theta_p^s + \phi_0)$$

$$\cdots 8.56$$

Some of the terms in equation (8.54) produce double frequency
components, the time average of which is zero. The net contribution
of the integral in equation (8.53) comes from $E_{t1}^{rs} H_{t1\theta}^{rr}$ and $E_{t1}^{rr} H_{t1\theta}^{rs}$ and
with respect to the rotor co-ordinate system, the value of the in-
tegral is

$$P_{av}^r = \omega_r L^{rs} I^s I^r \sin(\theta_p^r - \theta_p^s - \phi_0) \qquad \cdots 8.57$$

It may be readily shown that with respect to the stator system

$$P_{av}^s = \omega_s L^{sr} I^s I^r \sin(\theta_p^r - \theta_p^s - \phi_0)$$

$$\cdots 8.58$$

Equation (8.57) and (8.58) reveal that the power crossing the
air gap depends on the co-ordinate system such that

$$\frac{P_{av}^s}{P_{av}^r} = \frac{\omega_s}{\omega_r} \qquad \cdots 8.59$$

This difference arises due to the fact that with respect to the
rotor system, the machine is not doing any mechanical work; $P_{av}^r = T\omega_r$
is lost in the ohmic resistance. Whereas, with respect to the stator
system the machine is doing mechanical work at a rate $T\omega_m$ in addition

to supplying the ohmic loss. It may be noted that P^S_{av} is the true power crossing the air gap. It comes from the stator which is supplied from the connected power system.

8.8 P-T Relation

From equation (8.51), (8.52), (8.57) and (8.58) it follows that

$$\frac{P^r_{av}}{T^r_{em}} = \omega_r \quad \text{and} \quad \frac{P^S_{av}}{T^S_{em}} = \omega_s$$

In general it may be written that

$$\frac{P}{T} = \omega \qquad \qquad \ldots 8.60$$

Where, ω is the angular velocity of the field components relative to a co-ordinate system with respect to which the surface of integration to obtain P is stationary. ω may not be equal to the angular frequency of the time functions involved. For example, if the surface current distributions are sinusoidal functions of $n\theta$, instead of θ, then

$$\omega_s = n\omega_m + \omega_r$$

or,

$$\frac{\omega_s}{n} = \omega_m + \frac{\omega_r}{n}$$

Where, $\frac{\omega_s}{n}$ is the angular velocity of the field components with respect to the stator and $\frac{\omega_r}{n}$ is the angular velocity with respect to rotor. Evaluating the integrals for P and T it may be shown that

$$\frac{P^r_{av}}{T} = \frac{\omega_r}{n} \quad \text{and} \quad \frac{P^S_{av}}{T} = \frac{\omega_s}{n}$$

Finally, it may be noted that the relation involving P T and ω is obtained strictly from the concept of electromagnetic angular momentum flow in the air gap of the machine.

8.9 Electromagnetic Angular Momentum Velocity Across the Air

Gap of the Induction Machine

A quantity of some interest is the velocity of propagation of electromagnetic angular momentum across the air gap of the induction machine. This also requires a careful definition. By analogy with fluid flow, the electromagnetic angular momentum velocity may be defined as the ratio of electromagnetic angular momentum flux per unit area per unit time, divided by the electromagnetic angular momentum density. In the case of the induction motor with a short air gap length, time average values of these quantities are used and the electromagnetic angular momentum velocity is taken as the ratio of the time average electromagnetic angular momentum flux across the air gap to the mean stored electromagnetic angular momentum per unit air gap length, that is,

$$V_\omega = \frac{\int_S \vec{r} \times (M \cdot \vec{n}) dS \Big|_{av}}{\frac{1}{gc^2} \int_V (\vec{r} \times \vec{s}) dV \Big|_{av}} \qquad \ldots 8.61$$

Where the integration in the numerator is over a cylindrical surface in the air gap, g is the air gap length and the integration in the denominator is over the entire volume of the air gap.

From equation (8.52), at the steady state

$$\int_S \vec{r} \times (M \cdot \vec{n}) dS \Big|_{av} = L^{sr} I^s I^r \sin(\theta_p^r - \theta_p^s - \phi_o) \qquad \ldots 8.62$$

where

$$L^{sr} = \mu_o \pi \ell N_1^s N_1^r (\frac{c}{\omega_s}) X^{sr} \qquad \ldots 8.63$$

with

$$X^{sr} = \left(\frac{2c}{\omega_s}\right) \Big/ \left[J_1'(\omega_s b/c)\, Y_1'(\omega_s a/c) - Y_1'(\omega_s b/c)\, J_1'(\omega_s a/c)\right]$$

...8.64

For low frequencies and small air gap length

$$L^{sr} = \left(2\mu_o \pi\, l\, a\, b\, N_1^S N_1^r\right) \Big/ \left[(b/a) - (a/b)\right]$$

...8.65

From equation (6.12) the time average of electromagnetic

angular momentum storage is

$$G\omega = \frac{l}{c^2}\int_V (\vec{r} \times \vec{S})\, dV\,\Big|_{av} = \frac{l}{c^2}\int_a^b\!\!\int_0^{2\pi} r^2 S_\theta\, dr\, d\theta\,\Big|_{av}$$

...8.66

With respect to the stator co-ordinate system

$$S_\theta = E_{t1}^{ss} H_{t1r}^{ss} + E_{t1}^{ss} H_{t1r}^{sr} + E_{t1}^{sr} H_{t1r}^{ss} + E_{t1}^{sr} H_{t1r}^{sr}$$

...8.67

Expressing the field components in terms of the stator system,

S_θ may be obtained from equation (8.64). Substitution of S_θ in

equation (8.63) and the evalution of the indicated integral gives

$$G_\omega = \frac{1}{\omega_s}\mu_o \pi l\, N_1^S N_1^S I^S I^S \xi^S + \frac{1}{\omega_s}\mu_o \pi l\, N_1^r N_1^r I^r I^r \xi^r$$

$$+ \frac{2}{\omega_s}\mu_o \pi l\, N_1^S N_1^r I^S I^r \xi^{sr} \cos\left(\theta_P^S - \theta_P^r + \phi_o\right)$$

...8.68

where

$$\xi^S = \int_a^b r \left[\frac{\left[J_1'(\omega_s a/c) Y_1(\omega_s r/c) - Y_1'(\omega_s a/c) J_1(\omega_s r/c)\right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c)\right]}\right]^2 dr$$

...8.69

$$\xi^r = \int_a^b r \left[\frac{\left[J_1'(\omega_s b/c) Y_1(\omega_s r/c) - Y_1'(\omega_s b/c) J_1(\omega_s r/c)\right]}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c)\right]}\right]^2 dr$$

...8.70

and

$$\xi^{sr} = \int_a^b r \left[\frac{[J_1'(\omega_s a/c)\, Y_1(\omega_s r/c) - Y_1'(\omega_s a/c)\, J_1(\omega_s r/c)]}{[J_1'(\omega_s a/c)\, Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\, J_1'(\omega_s b/c)]} \right] \cdot$$

$$\left[\frac{[J_1'(\omega_s b/c)\, Y_1(\omega_s r/c) - Y_1'(\omega_s b/c)\, J_1(\omega_s r/c)]}{[J_1'(\omega_s a/c)\, Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\, J_1'(\omega_s b/c)]} \right] dr$$

...8.71

For low frequencies and small air gap

$$\xi^s = \int_a^b \frac{\omega_s^2 b^4}{c^2 (b^2-a^2)^2}\, r^3 \left(1 + \frac{a^2}{r^2}\right)^2 dr$$

$$= \frac{\omega_s^2 b^4}{c^2 d^2}\, \xi_1^s$$

...8.72

$$\xi^r = \int_a^b \frac{\omega_s^2 a^4}{c^2 (b^2-a^2)^2}\, r^3 \left(1 + \frac{b^2}{r^2}\right)^2 dr$$

$$= \frac{\omega_s^2 a^4}{c^2 d^2}\, \xi_1^r$$

...8.73

and

$$\xi^{sr} = \int_a^b \frac{\omega_s^2 a^2 b^2}{c^2 (b^2-a^2)^2}\, r^3 \left(1 + \frac{a^2}{r^2}\right)\left(1 + \frac{b^2}{r^2}\right) dr$$

...8.74

$$= \frac{\omega_s^2 a^2 b^2}{c^2 (b^2-a^2)^2}\, \xi_1^{sr}$$

where

$$\xi_1^s = \frac{b^4}{4} + b^2 a^2 - \frac{5}{4} a^4 + a^4 \ln b/a$$

...8.75

$$\xi_1^r = \frac{5}{4} b^4 - b^2 a^2 - \frac{a^4}{4} + b^4 \ln b/a$$

...8.76

$$\xi_1^{sr} = \frac{3}{4} b^4 - \frac{3}{4} a^4 + b^2 a^2 \ln b/a$$

...8.77

and $\qquad d = b^2 - a^2$ $\qquad\qquad\qquad$ ...8.78

Substituting equations (8.62) and (8.68) in equation (8.61) one obtains

$$\upsilon_\omega = \frac{\left[c g N_1^S N_1^r I^S I^r \times {}^{sr} \sin(\theta_P^r - \theta_P^S - \phi_0)\right]}{\left[(N_1^S I^S)^2 \xi^S + (N_1^r I^r)^2 \xi^r + 2 N_1^S N_1^r I^S I^r \xi^{sr} \cos(\theta_P^r - \theta_P^S - \phi_0)\right]}$$

...8.79

where $\times^{sr}$, ξ^S, ξ^r and ξ^{sr} are given by equations (8.64), (8.69), (8.70) and (8.71) respectively.

For low frequencies and short air gap

$$\upsilon_\omega = \frac{\left[2 c^2 g N_1^S N_1^r I^S I^r a^2 b^2 d \sin(\theta_P^r - \theta_P^S - \phi_0)\right]}{\omega_s \left[(N_1^S I^S)^2 b^4 \xi_1^S + (N_1^r I^r)^2 a^4 \xi_1^r + 2 N_1^S N_1^r I^S I^r a^2 b^2 \xi_1^{sr} \cos(\theta_P^r - \theta_P^S - \phi_0)\right]}$$

...8.79A

where ξ_1^S, ξ_1^r, ξ_1^{sr} and d are given by equations (8.75), (8.76), (8.77) and (8.78) respectively.

8.10 Energy Velocity

As in the case of electromagnetic angular momentum, one might enquire as to the velocity of propagation of electromagnetic energy[49] across the air gap of the induction machine. For a short air gap length a definition of the energy velocity υ_E, similar to that of the angular momentum velocity υ_ω, gives

$$\upsilon_E = \frac{-\int_S (\vec{E} \times \vec{H}) dS \big|_{av}}{\frac{1}{g} \int_V (\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}) dv \big|_{av}} \qquad\qquad ...8.80$$

Where the integral in the numerator is the time average electromagnetic power flux across the air gap and the integral in the denominator is the time average electromagnetic energy stored in the entire volume of the air gap.

The steady state power flux is obtained from equation (8.58) such that

$$P_{av}^s = -\int_S (\vec{E} \times \vec{H}) dS \Big|_{av} = \omega_s L^{sr} I^s I^r \sin(\theta_P^r - \theta_P^s - \phi_o)$$

...8.81

If W_E is the average electric energy stored, it may be shown that

$$W_E = \int_V \frac{1}{2}(\vec{E}.\vec{D}) dV \Big|_{av}$$

$$= \frac{1}{2}\mu_o \pi \ell (N_1^s I^s)^2 \xi^s + \frac{1}{2}\mu_o \pi \ell (N_1^r I^r)^2 \xi^r$$

$$+ \mu_o \pi \ell N_1^s N_1^r I^s I^r \xi^{sr} \cos(\theta_P^r - \theta_P^s - \phi_o)$$

...8.82

which is equation (8.68) multiplied by ω_s. For low frequencies and short air gap,

$$W_E = (\frac{\omega_s^2}{2c^2d^2})\mu_o \pi \ell (N_1^s I^s)^2 b^4 \xi_1^s + (\frac{\omega_s^2}{2c^2d^2})\mu_o \pi \ell (N_1^r I^r)^2 a^4 \xi_1^r$$

$$+ \frac{\omega_s^2}{c^2d^2}\mu_o \pi \ell N_1^s N_1^r I^s I^r b^2 a^2 \xi_1^{sr} \cos(\theta_P^r - \theta_P^s - \phi_o)$$

...8.83

where the quanties ξ_1^s, ξ_1^r, ξ_1^{sr} and d are given by equations (8.75), (8.76), (8.77) and (8.78) respectively.

If W_m is the average magnetic energy stored, it may be readily show that

$$W_m = \int_V \frac{1}{2}(\vec{H}.\vec{B}) dV \Big|_{av}$$

$$= \frac{1}{2}\mu_o \pi \ell (N_1^s I^s)^2 (\eta_1^s + \eta_2^s)$$

$$+ \frac{1}{2}\mu_0 \pi l \left(N_1^r I^r\right)^2 \left(\eta_1^r + \eta_2^r\right)$$

$$+ \mu_0 \pi l \; N_1^s N_1^r I^s I^r \left(\eta_1^{sr} + \eta_2^{sr}\right)\cos\left(\theta_P^r - \theta_P^s - \phi_0\right)$$

$$\dots 8.84$$

where

$$\eta_1^s = \frac{c^2}{\omega_s^2}\int_a^b \frac{1}{r}\left[\frac{J_1'(\omega_s a/c)\,Y_1(\omega_s r/c) - Y_1'(\omega_s a/c)\,J_1(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]^2 dr \quad \dots 8.85$$

$$\eta_2^s = \int_a^b r\left[\frac{J_1'(\omega_s a/c)\,Y_1'(\omega_s r/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]^2 dr \quad \dots 8.86$$

$$\eta_1^r = \frac{c^2}{\omega_s^2}\int_a^b \frac{1}{r}\left[\frac{J_1'(\omega_s b/c)\,Y_1(\omega_s r/c) - Y_1'(\omega_s b/c)\,J_1(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]^2 dr \quad \dots 8.87$$

$$\eta_2^r = \int_a^b r\left[\frac{J_1'(\omega_s b/c)\,Y_1'(\omega_s r/c) - Y_1'(\omega_s b/c)\,J_1'(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]^2 \cdot dr \quad \dots 8.88$$

$$\eta_1^{sr} = \frac{c^2}{\omega_s^2}\int_a^b \frac{1}{r}\left[\frac{J_1'(\omega_s a/c)\,Y_1(\omega_s r/c) - Y_1'(\omega_s a/c)\,J_1(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]\cdot$$

$$\left[\frac{J_1'(\omega_s b/c)\,Y_1'(\omega_s r/c) - Y_1'(\omega_s b/c)\,J_1(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right] dr \quad \dots 8.89$$

$$\eta_2^{sr} = \int_a^b r\left[\frac{J_1'(\omega_s a/c)\,Y_1'(\omega_s r/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right]\cdot$$

$$\left[\frac{J_1'(\omega_s b/c)\,Y_1'(\omega_s r/c) - Y_1'(\omega_s b/c)\,J_1'(\omega_s r/c)}{J_1'(\omega_s a/c)\,Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\,J_1'(\omega_s b/c)}\right] dr \quad \dots 8.90$$

For low frequencies and short air gap length

$$\eta_1^s = \frac{b^4}{d^2}\left[\frac{b^2}{2} - \frac{a^4}{2b^2} + 2a^2 \ln b/a\right] \quad \dots 8.91$$

$$\eta_2^s = \frac{b^4}{d^2}\left[\frac{b^2}{2} - \frac{a^4}{2b^2} - 2a^2 \ln b/a\right] \quad \dots 8.92$$

$$\eta_1^r = \frac{a^4}{d^2}\left[\frac{b^4}{2a^2} - \frac{a^2}{2} + 2b^2\ln b/a\right] \qquad \ldots 8.93$$

$$\eta_2^r = \frac{a^4}{d^2}\left[\frac{b^4}{2a^2} - \frac{a^2}{2} - 2b^2\ln b/a\right] \qquad \ldots 8.94$$

$$\eta_1^{sr} = \frac{b^2 a^2}{d^2}\left[b^2-a^2 + (b^2+a^2)\ln b/a\right] \qquad \ldots 8.95$$

$$\eta_2^{sr} = \frac{b^2 a^2}{d^2}\left[b^2-a^2 - (b^2+a^2)\ln b/a\right] \qquad \ldots 8.96$$

Hence
$$v_E = \frac{P_{av}^s}{\frac{1}{g}(W_E + W_m)} \qquad \ldots 8.97$$

Where, P_{av}^s, W_E and W_m are defined by equations (8.81), (8.82) and (8.84) respectively.

For low frequencies W_E may be neglected in comparison with W_m. Then for short air gap

$$v_E = \frac{\left[4g\,\omega_s\, N_1^s N_1^r I^s I^r\, b^2 a^2 d \sin(\theta_p^r - \theta_p^s - \beta)\right]}{\left[(N_1^s I^s)^2 b^4 \eta_3^s + (N_1^r I^r)^2 a^4 \eta_3^r + 2 N_1^s N_1^r I^s I^r b^2 a^2 \eta_3^{sr} \cos(\theta_p^r - \theta_p^s - \beta)\right]}$$
$$\ldots 8.97A$$

where
$$\eta_3^s = \left(b^2 - \frac{a^4}{b^2}\right) \qquad \ldots 8.98$$

$$\eta_3^r = \left(\frac{b^4}{a^2} - a^2\right) \qquad \ldots 8.99$$

and $\qquad \eta_{3}^{s\gamma} = 2(b^2 - a^2)$ $\qquad\qquad\qquad$ ...8.100

8.11 Discussion and Numerical Results

(A) Discussion on the Exact Solution:- A detailed field theory analysis of an induction motor has been developed to study the steady state production of torque in the rotor in terms of transfer of electromagnetic angular momentum across the air gap. For specified surface current distributions, the air gap field quantities are obtained from the exact solution of Maxwell's equations. It may be noted that these fields have the form of electromagnetic waves. An examination of the nature of these waves may be helpful to clarify the physical implications involved in the analysis. Since the general description of the family of waves remains the same whether they are referred to the stator or rotor co-ordinate system, the discussion will be limited to the latter case.

Consider equations (8.27) and (8.48). These are the electromagnetic fields produced by the stator current acting alone. It is interesting to note that a point (r, θ, z) the components of the magnetic field intensity form an elliptically polarized vector. In a plane z = constant, the electromagnetic fields result in a circularly polarized wave. In the three dimensional space of the air gap the field pattern is a circularly polarized standing surface wave. Being a standing wave it gives storage of energy without propagation in any direction. It may be seen that due to the

interaction of the z-directed E-field and the radial component of the
H-field of the circularly polarized standing wave there can be a net
electromagnetic angular momentum storage in the air gap.

Consider equations (8.26) and (8.49), which give the field quan-
tities produced by the rotor current acting alone. The field pattern
is of an identical form, namely a circularly polarized standing
surface wave, as that produced by the stator current except for a
phase displacement and a difference in magnitude. Acting alone this
wave gives a storage of electromagnetic energy and angular momentum
without any propagation.

Consider now the combination of the stator-and rotor-induced
circularly polarized waves. The interaction between the E-field of
each of these waves and the tangential component of the H-field of
the other results in a net flow of energy radially inward. Similarly,
the radial component of the H-field of each wave reacts with the
tangential component of the H-field of the other to give a net el-
ectromagnetic angular momentum flow. The electromagnetic energy and
angular momentum are absorbed on the rotor surface at continuous
rates so that the flow of these quantities is maintained

The concept of electromagnetic angular momentum flow can be
further enhanced by considering a machine without any iron but other-
wise the same as that discussed so far. It may be shown that the
field pattern produced by the stator or rotor current is a circularly
polarized standing wave inside it (stator or rotor) and a radiation
field outside with waves spiralling outwards. Thus, inside the

rotor there is a circularly polarized standing wave made up of contributions from the stator and rotor currents. This wave gives no propagation of electromagnetic energy and angular momentum in any direction.

In the air gap between the stator and rotor there is a combination of the stator-induced standing wave and the rotor-induced outward spiralling propagating wave. The combined field cause a net flow of energy and electromagnetic angular momentum inward. Outside the stator there is a resulting outgoing spiralling wave made up of the sum of the component waves produced by the stator and rotor. An interesting fact is that

$$\left.\int_S \vec{r} \times (M.\vec{n})ds\right|_{av}$$

does not vanish when taken over a surface just outside the stator and as a result the mechanical torque on the rotor is not exactly equal to the mechanical torque on the stator. The reason for this is that the stator is radiating energy and electromagnetic angular momentum outwards into space. In the actual machine the presence of an iron circuit outside the stator current sheet prevents (when $\mu_r \to \infty$) the outward flow of electromagnetic angular momentum, thus making the stator and rotor torque equal in magnitude. The main effect of iron in the machine is an increase of the air gap magnetic flux by a large factor, thereby increasing the torque and power transfer.

(B) Discussion on Quasi-static Approximations:- As noted in Chapter 7, the quasi-static field quantities may be directly obtained by using the small argument approximations of the Bessel functions. At power frequencies, even taking the mean radius of the machine as great as 1 meter, the argument is of the order of 10^{-6}. From the series expansions of the Bessel functions it may be shown that the approximations

$$\left. \begin{array}{l} J_1(z) = \dfrac{z}{2} \\[2mm] Y_1(z) = -\dfrac{2}{\pi z} \end{array} \right\} \quad z = 10^{-6}$$

give the values of these functions and their derivatives correct to one part in at least 10^{10}. Hence, it may be stated that in magnitude the quasi-static field values are correct in about the same proportion.

It may also be noted that the momentum concept may be used to evaluate the torque even if the electromagnetic field quantities are calculated on the basis of quasi-static approximation. However, in the ironless machine with such an approximation

$$\int \vec{r} \times (M.\vec{n}) dS$$

vanishes outside the stator showing no radiation of energy and electromagnetic angular momentum outwards from the machine.

This truly remarkable agreement between the exact and approximate solutions only refers to the steady state operation of the machine. It will be shown in the next chapter that if a sudden change is made, for example in the value of the rotor resistance,

the quasi-static approximation is inadequate to discribe the forces
which operate during the transition from one steady state to
another. A change in the amount of electromagnetic angular mom-
entum stored in the air gap of the machine must take place and
there will be torque impulses on the stator and rotor windings
to supply this change.

(B)   Numerical Results

The steady state machine characteristics such as torque-
slip relationships with rotor impedance control may be determined
by simple manipulation of the relevent equations obtained in
this chapter. The calculated results should agree with experi-
mental ones. Numerical calculations are made for a sleeve rotor
machine with two rotor resistance values, to study energy and
angular momentum velocities as functions of air gap length. The
rotor is taken as stationary and a change in rotor surface res-
istivity implies a change in sleeve thickness.

Specifications

$$a \ = \ 1m \ = \ \text{Rotor Radius}$$

$$R_r \ = \ 1.72(10^{-5}) \ \text{or} \ 1.72(10^{-4}) \ \text{ohm/m}^2$$

$$= \ \text{Rotor Surface Resistivity}$$

$$v^{11} \ = \ \text{velocity for } R_r = 1.72(10^{-5}) \ \text{ohm/m}^2$$

$$v^{12} \ = \ \text{velocity for } R_r = 1.72(10^{-4}) \ \text{ohm/m}^2$$

The results are shown as graphs in Fig. (8.1)

It may be noticed that across the air gap of an induction
machine the average electromagnetic momentum velocity is higher
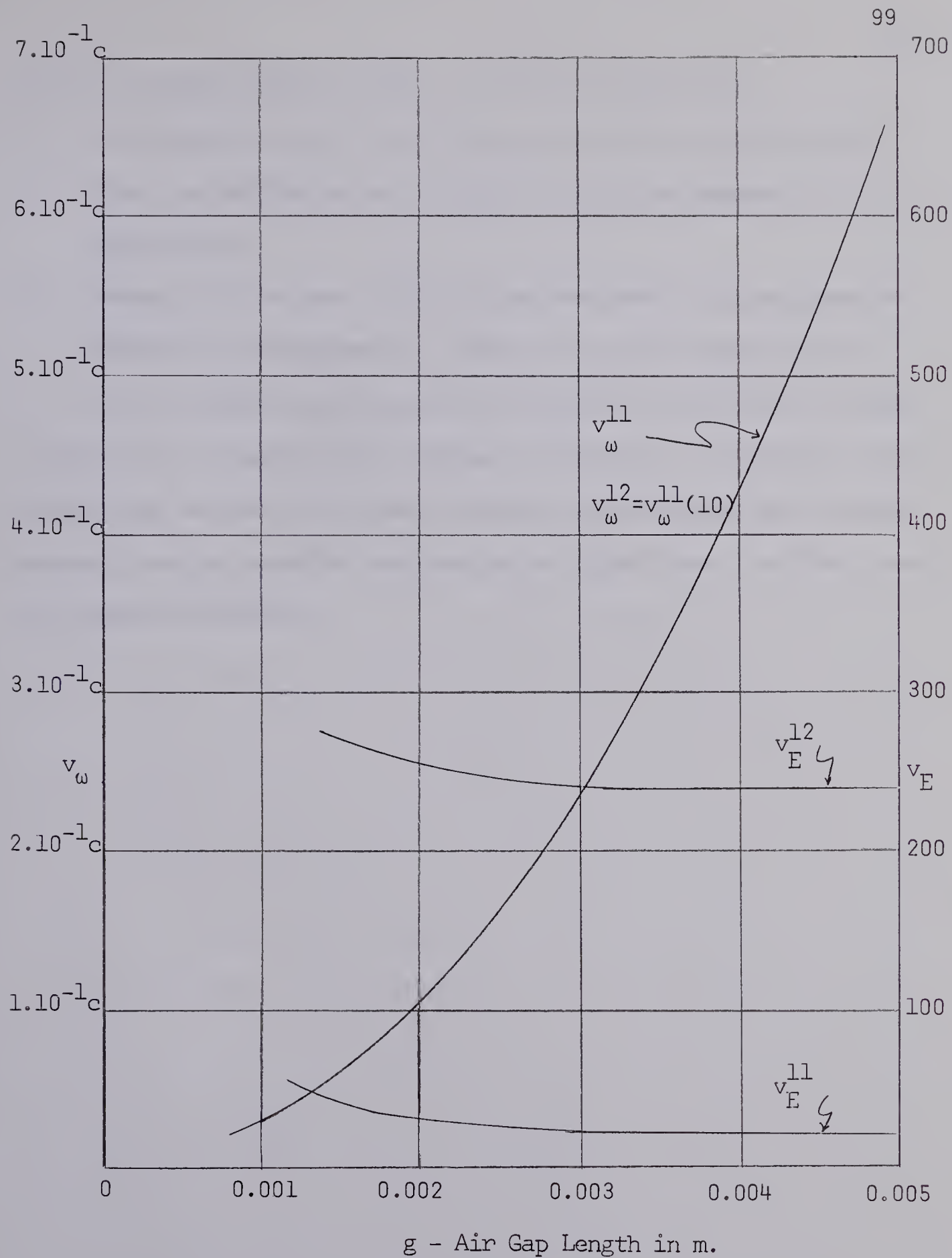
Fig. 8.1 – Energy and Electromagnetic Angular Momentum
Velocities Versus Air Gap Length

than the average energy velocity. The main reasons are

(1)   The equations (8.79) or (8.79A) and (8.97) or (8.97A) giving
       these velocities do not account for all electromagnetic field
       quantities.

(2)   Because of the term $1/c^2$ the electromagnetic angular momentum
       density is a much smaller quantity than the energy density.

       For the same air gap length both v_ω and v_E have shown increased
values with increasing rotor surface resistivity. The physical impli-
cations may be that for higher resistance values energy and electro-
magnetic angular momentum are absorbed at higher rates, so they travel
with faster velocities.

CHAPTER 9

ANGULAR MOMENTUM FLOW

AT TRANSIENT STATE

9.1 Introduction

When a sudden change is made in a system, the solution of the

system differential equations must satisfy the initial conditions.

The solution by the method of Laplace Transform takes care of that

by finding out the free response consisting of all the terms (known

as transient terms) which are contributed by the natural modes, i.e.

the roots of the characteristic equation together with the forced

response consisting of the terms contributed by the poles of the

driving functions. (The latter terms are known as the steady state

solutions when the driving functions are periodic.)

The nature of electromagnetic angular momentum flow and the

method of its study remains basically the same no matter how the

transient is instigated. The problem will be illustrated by using

a simple transient. The transient will be created by assuming that

a two-pole (space fundamental only present), two-phase machine is

operating at steady state with rotor open circuited and constrained

to remain stationary. At t=0 a sudden change is made in the value

of the rotor resistance (open circuit to closed circuit). The

stator is supplied from a "Constant Current Source". If the

stator is supplied from a "Constant Voltage Source" the solution

of the problem will involve only more algebraic manipulations.

9.2 Initial Conditions and Resultant Vector Potentials

Let the stator surface current densities (amp/length) corres-
ponding to the currents (amp) supplied from the "Constant Current
Source" be specified as

$$i^s_{1a}(t) = \hat{I}^s_1 \cos \omega_s t \qquad \dots 9.1$$

$$i^s_{1b}(t) = \hat{I}^s_1 \sin \omega_s t \qquad \dots 9.2$$

The ideal source is taken to have an infinite impedance, hence, these
current densities will be maintained during the transient conditions
and the corresponding vector potentials may be obtained from equa-
tions (8.15) and (8.16) by substituting I^s_1 for $N^s_1 I^s$ and putting
$\theta^s_p = 0$ such that

$$A^s_{a1}(r,\theta,t) = \mu_0 \hat{I}^s_1 \frac{c}{\omega_s} X^{ss}_1 (r,\omega_s) \sin\theta \cos \omega_s t \qquad \dots 9.3$$

$$A^s_{b1}(r,\theta,t) = -\mu_0 \hat{I}^s_1 \frac{c}{\omega_s} X^{ss}_1 (r,\omega_s) \cos\theta \sin \omega_s t \qquad \dots 9.4$$

where

$$X^{ss}_1(r,\omega_s) = \frac{[J'_1(\omega_s a/c) Y_1(\omega_s r/c) - Y'_1(\omega_s a/c) J_1(\omega_s r/c)]}{[J'_1(\omega_s a/c) Y'_1(\omega_s b/c) - Y'_1(\omega_s a/c) J'_1(\omega_s b/c)]} \qquad \dots 8.14$$

$A^s_{a1}(r,\theta,t)$ and $A^s_{b1}(r,\theta,t)$ are known functions in the time domain.

Initially the vector potential and its derivatives arising from
the rotor current are zero. Furthermore, as implied in section (8.5),

the electric field intensities at the rotor surface due to stator currents vary as $\sin\theta$ and $\sin(\theta - \pi/2)$ and these are the only external driving functions for the rotor current. Hence, for the same co-ordinate system a component of rotor current will vary as $\sin\theta$ and the other as $\sin(\theta - \pi/2)$. Denoting the current densities corresponding to the above two components of rotor current as $\bar{I}^r_{1a}$ and $\bar{I}^r_{1b}$ and setting the initial conditions zero, the s-domain equations of the rotor vector potentials may be written from equations (4.15) and (4.16) as

$$\bar{A}^r_{a1}(r,\theta,s) = \mu_o \bar{I}^r_{1a}(s) \, \frac{c}{s} \, \bar{X}^{rr}_1 \, (r,s) \, \sin\theta \qquad \ldots 9.5$$

$$\bar{A}^r_{b1}(r,\theta,s) = \mu_o \bar{I}^r_{1b}(s) \, \frac{c}{s} \, \bar{X}^{rr}_1(r,s) \sin(\theta - \pi/2) \qquad \ldots 9.6$$

where

$$\bar{X}^{rr}_1(r,s) = \frac{\left[K'_1(sb/c)I_1(sr/c) - I'_1(sb/c)K_1(sr/c) \right]}{\left[K'_1(sa/c)I'_1(sb/c) - I'_1(sa/c)K'_1(sb/c) \right]} \qquad \ldots 8.20$$

As in section (8.5) $\bar{I}^r_{1a}(s)$ and $\bar{I}^r_{1b}(s)$ may be evaluated by equating the resistive drop along the rotor length for each of this current to the electric intensity $\bar{E}_z(s)$ which is the sum of the intensities produced by the stator and rotor currents of similar phases such that

$$R_r \bar{I}^r_{1a}(s)\sin\theta = \bar{E}^s_{a1}(s) + \bar{E}^r_{a1}(s) \qquad \ldots 9.7$$

$$R_r \bar{I}_{1b}^r(s) \sin(\theta - \pi/2) = \bar{E}_{b1}^s(s) + \bar{E}_{b1}^r(s) \qquad \ldots 9.8$$

where

$$R_r = \text{Rotor Surface Resistivity,}$$

$$\bar{E}_{a1}^s(s) = \frac{\omega_s^2 L I_1^s}{(s^2 + \omega_s^2)} \sin\theta \qquad \ldots 9.9$$

$$\bar{E}_{b1}^s(s) = - \frac{s \omega_s L I_1^s}{s^2 + \omega_s^2} \sin(\theta - \pi/2) \qquad \ldots 9.10$$

with

$$L = -\mu_0 \frac{c^2}{\omega_s^2} \frac{2}{\pi a} \frac{1}{\left[J_1'(\omega_s a/c) Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c) \right]} \qquad \ldots 9.11$$

and

$$\bar{E}_{a1}^r(s) = -\mu_0 c \, \bar{I}_{1a}^r(s) \, \bar{X}_1^{rr}(s) \sin\theta \qquad \ldots 9.12$$

$$\bar{E}_{b1}^r(s) = -\mu_0 c \, \bar{I}_{1b}^r(s) \, \bar{X}_1^{rr}(s) \sin(\theta - \pi/2) \qquad \ldots 9.13$$

with

$$\bar{X}_1^{rr}(s) = \frac{\left[K(sb/c) I_1(sa/c) - I_1'(sb/c) K_1(sa/c) \right]}{\left[K_1'(sa/c) I_1'(sb/c) - I_1'(sa/c) K_1'(sb/c) \right]} \qquad \ldots 9.14$$

Hence,

$$\bar{I}_{1a}^r(s) = \frac{\omega_s^2 L I_1^s}{\left[R_r + \mu_0 c \, \bar{X}_1^{rr}(s) \right] (s^2 + \omega_s^2)} \qquad \ldots 9.15$$

$$\bar{I}^{\,r}_{1b}(s) = - \frac{s\,\omega_s\,L\,I_1^s}{(s^2 + \omega_s^2)\left[R_r + \mu_o c\,\bar{X}_1^{rr}(s)\right]} \qquad \ldots 9.16$$

Putting these current densities in the vector potential equations (9.5) and (9.6) one obtains

$$\bar{A}^{r}_{a1}(r,\theta,s) = \mu_o\,\frac{c}{s}\,\frac{\omega_s^2\,L\,I_1^s\,\bar{X}_1^{rr}(r,s)\,\sin\theta}{(s^2 + \omega_s^2)\left[R_r + \mu_o c\,\bar{X}_1^{rr}(s)\right]} \qquad \ldots 9.17$$

$$\bar{A}^{r}_{b1}(r,\theta,s) = \mu_o c\,\frac{\omega_s\,L\,I_1^s\,\bar{X}_1^{rr}(r,s)\,\cos\theta}{(s^2 + \omega_s^2)\left[R_r + \mu_o c\,\bar{X}_1^{rr}(s)\right]} \qquad \ldots 9.18$$

9.3 Time Solution for the Vector Potentials

The time functions may be obtained from the Inversion Theorem,

$$A^{r}_{a1}(r,\theta,t) = \frac{C_o\,\sin\theta}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{\omega_s}{s}\,\frac{\bar{X}_1^{rr}(r,s)\,e^{st}\,ds}{(s^2 + \omega_s^2)\left[R_r + \mu_o c\,\bar{X}_1^{rr}(s)\right]} \qquad \ldots 9.19$$

$$A^{r}_{b1}(r,\theta,t) = \frac{C_o\,\cos\theta}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\frac{\bar{X}_1^{rr}(r,s)\,e^{st}\,ds}{(s^2 + \omega_s^2)\left[R_r + \mu_o c\,\bar{X}_1^{rr}(s)\right]} \qquad \ldots 9.20$$

where

$$C_o = \mu_o c\,\omega_s\,L\,I_1^s \qquad \ldots 9.21$$

The integrands are single valued functions of s. Near the origin

the functions $\bar{X}_1^{rr}(r,s)$ and $\bar{X}_1^{rr}(s)$ behave as

$$\lim_{S \to o} \bar{X}_1^{rr}(r,S) = \frac{sa}{c} \frac{[r/b + b/r]}{[b/a - a/b]}$$

$$\lim_{S \to o} \bar{X}_1^{rr}(s) = \frac{sa}{c} \frac{[a/b + b/a]}{[b/a - a/b]}$$

Hence the integrands do not have any pole at the origin. The poles are at

$$S = \pm j\omega_s \qquad \text{(corresponds to the driving function)}$$

and at

$$S = -\frac{R_r}{L_r}, \quad -d_\nu \pm j\beta_\nu \qquad \text{(corresponds to the transient terms)}$$

The latter poles are the roots of $R_r + \mu_o c \bar{X}_1^{rr}(s) = o$ and these are calculated in Appendix (3).

The first order approximation shown in equations (A3.38) and (A3.34) gives

$$L_r = \mu_o a \frac{[a/b + b/a]}{[b/a - a/b]} \qquad \qquad \ldots 9.22$$

$$d_\nu = -\frac{c}{2(b-a)} \ln \frac{[1 - R_r/\mu_o c]}{[1 + R_r/\mu_o c]} \qquad \ldots 9.23$$

$$\beta_\nu = \frac{\Pi c}{2} \frac{[1 + 2\nu]}{[b - a]} \quad , \quad \nu = o,1,2,\ldots$$

$$\ldots 9.24$$
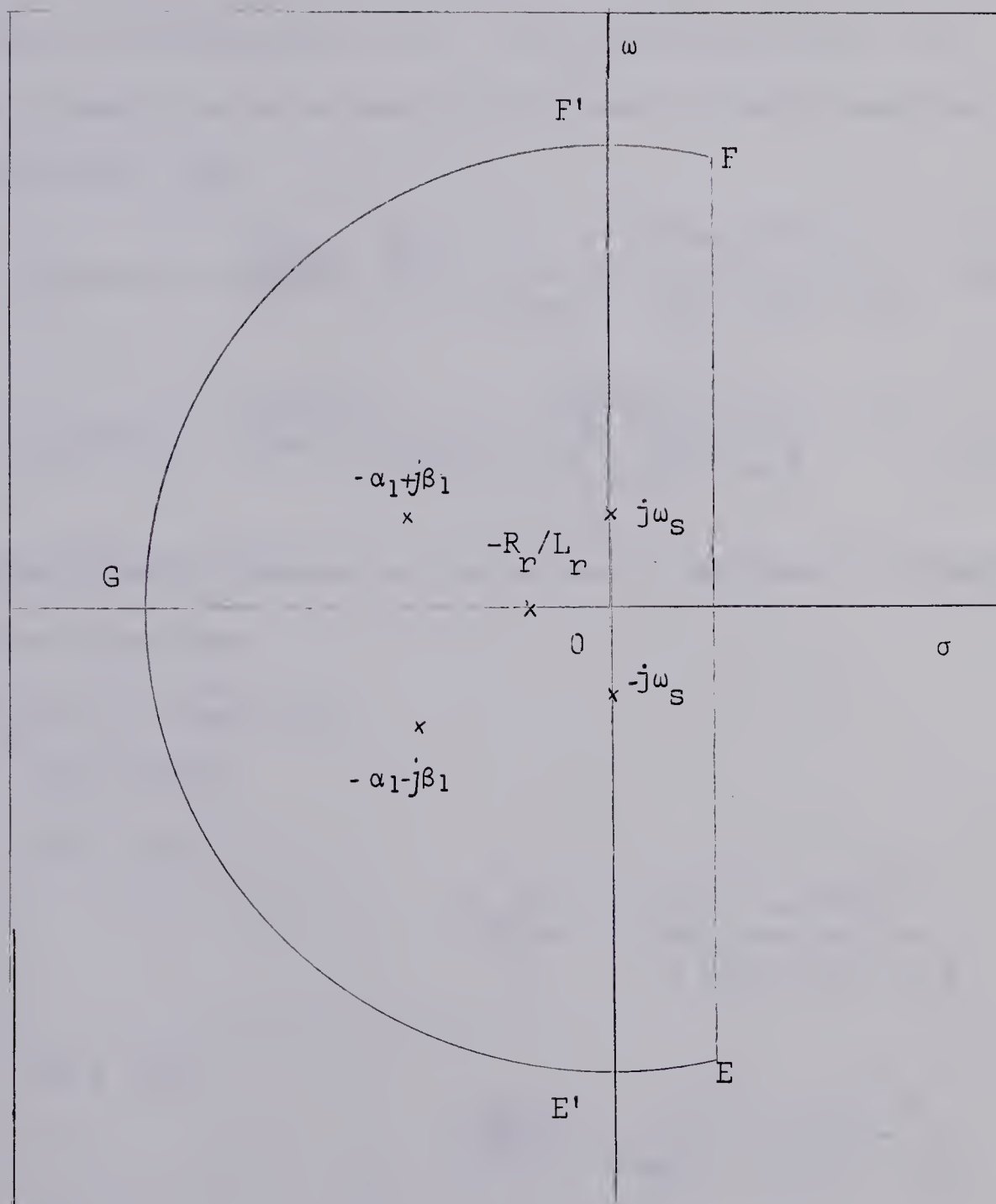
Fig. 9.1 - Contour of Integration

Consider the integrals taken over the closed contour of Fig. 9.1. The circle of radius $R = \frac{\pi c \nu}{b-a}$ does not pass through any pole of the integrands. In appendix (4), it is proved that as $\nu \to \infty$, the integrals over the arc FGE tends to zero. Thus in the limit as $\nu \to \infty$, the line integrals may be replaced by the closed integrals over the contour EFGE such that

$$A^r_{al}(r,\theta,t) = \frac{C_o \sin\theta}{2\pi j} \oint \frac{\omega_s}{s} \frac{\overline{X}^{rr}_1(r,s)e^{st}\, ds}{(s^2+\omega_s^2)\left[R_r+\mu_o c\overline{X}^{rr}_1(s)\right]} \quad \ldots 9.25$$

$$A^r_{bl}(r,\theta,t) = \frac{C_o \cos\theta}{2\pi j} \oint \frac{\overline{X}^{rr}_1(r,s)e^{st}\, ds}{(s^2+\omega_s^2)\left[R_r+\mu_o c\overline{X}^{rr}_1(s)\right]} \quad \ldots 9.26$$

Cauchy's Residue Theorem may now be used to evaluate the integrals of the above equations.

Evaluation of $A^r_{al}(r,\theta,t)$

(A) The residue

at $s = j\omega_s$

$$= \frac{C_o \sin\theta}{2\pi j} \frac{-jX^{rr}_1(r,\omega_s)e^{j\omega_s t}}{2\omega_s\left[R_r+j\mu_o cX^{rr}_1(\omega_s)\right]}$$

at $s = -j\omega_s$

$$= \frac{C_o \sin\theta}{2\pi j} \frac{jX^{rr}_1(r,\omega_s)e^{-j\omega_s t}}{2\omega_s\left[R_r - j\mu_o cX^{rr}_1(\omega_s)\right]}$$

and at $s = -\frac{R_r}{L_r}$

$$= \frac{C_o \sin\theta}{2\pi j} \frac{\omega_s}{c} \frac{L_{ra}\left[r/b + b/r\right]e^{-\frac{R_r t}{L_r}}}{\left[(R_r)^2+(\omega_s L_r)^2\right]\left[b/a-a/b\right]}$$

where

$X_1^{rr}(r,\omega_s)$ is given by equation (8.23) with ω_r replaced by ω_s; $X_1^{rr}(\omega_s)$ is given by the same equation with ω_r replaced by ω_s and r replaced by a.

(B) The residues at $s = -\alpha_\nu \pm j\beta_\nu$

The first order approximation will be used. From appendices (3) and (4) it follows that for large values of s

$$\frac{\bar{X}_1^{rr}(r,s)}{[R_r + \mu_o c \bar{X}_1^{rr}(s)]} = \sqrt{\frac{a}{r}} \cdot \frac{\cosh[(b-r)s/c]}{R_r \sinh[(b-a)s/c + \mu_o \cosh[(b-a)s/c]}$$

Also for large values of s

$$s(s^2 + \omega_s^2) = s^2$$

Hence the integrand

$$\frac{\omega_s \bar{X}_1^{rr}(r,s)e^{st}}{s(s^2 + \omega_s^2)[R_r + \mu_o c \bar{X}_1^{rr}(s)]} = \sqrt{\frac{a}{r}} \cdot \frac{\omega_s \cosh[(b-a)s/c]e^{st}}{s^3[R_r \sinh[(b-a)s/c] + \mu_o c \cosh[(b-a)s/c]}$$

The residue will be evaluated by writing the integrand as a ratio $p(s)/q'(s_o)$

$$p(s) = \sqrt{\frac{a}{r}} \cdot \frac{\omega_s \cosh[b-r)s/c]e^{st}}{s^3}$$

Since

$$|\alpha_\nu| \ll |\beta_\nu|$$

$$p(-\alpha_\nu + j\beta_\nu) = \sqrt{\frac{a}{r}} \cdot \frac{\omega_s \cos[-\frac{b-r}{b-a} \frac{\pi}{2}(1+2\nu)]}{-j\beta_\nu^3} e^{-\alpha t + j\beta_\nu t}$$

$$P(-\alpha_\nu - j\beta_\nu) = \sqrt{\frac{a}{r}} \; \frac{\omega_s \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right]}{j\beta_\nu^3} e^{-\alpha_\nu t - j\beta_\nu t}$$

$$q(s) = \left[R_r \sinh[(b-a)s/c] + \mu_0 c \cosh[(b-a)s/c] \right]$$

$$q'(s) = \left[R_r [(b-a)/c] \cosh[(b-a)s/c] + \mu_0(b-a) \sinh[(b-a)s/c] \right]$$

Now,

$$\cosh\left[\pm j\left(\frac{b-a}{c}\right)\beta_\nu\right] = 0$$

and

$$\sinh\left[\pm j\left(\frac{b-a}{c}\right)\beta_\nu\right] = \pm j \sin(1+2\nu)\frac{\pi}{2}$$

$$= \pm j, \quad \nu = 0, 2, 4, \cdots$$

$$= \mp j, \quad \nu = 1, 3, 5, \cdots$$

Hence

$$q'(\alpha_\nu \pm \beta_\nu) = \pm j\mu_0 (b-a), \quad \nu \text{ even-including zero}$$

$$= \mp j\mu_0 (b-a), \quad \nu \text{ odd}$$

Therefore, the residue at $s = -\alpha_\nu + j\beta_\nu$

$$(-1)^\nu \frac{C_0 \sin\theta}{2\pi j} \sqrt{\frac{a}{r}} \cdot \frac{\omega_s \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right]}{\mu_0(b-a)\beta_\nu^3} e^{-\alpha_\nu t + j\beta_\nu t}$$

and that at $s = -\alpha_\nu - j\beta_\nu$ is

$$(-1)^\nu \frac{C_0 \sin\theta}{2\pi j} \sqrt{\frac{a}{r}} \frac{\omega_s \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right]}{\mu_0(b-a)\beta_\nu^3} e^{-\alpha_\nu t - j\beta_\nu t}$$

From Cauchy's Residue Theorem

$$A_{a1}^r(r,\theta,t) = 2\pi j \sum \text{Residues}$$

Summing up the residues

$$A_{a1}^r(r,\theta,t) = \left[\frac{C_0}{\omega_s} \frac{X_1^{rr}(r,\omega_s)\sin\theta \sin(\omega_s t - \psi_r)}{\sqrt{(R_r)^2 + (\mu_0 c X_1^{rr}(\omega_s))^2}} \right.$$

$$+ \frac{C_0 \omega_s}{c} \frac{L_r a \sin\theta [r/b + b/r]}{[(R_r)^2 + (\omega_s L_r)^2][b/a - a/b]} e^{-\frac{R_r}{L_r}t}$$

$$\left. + \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2C_0 \omega_s \sin\theta}{\mu_0(b-a)\beta_\nu^3} \sqrt{\frac{a}{r}} \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right] e^{-\alpha_\nu t} \cos\beta_\nu t \right]$$

...9.27

where,

$$\psi_r = \tan^{-1} \mu_0 c X_1^{rr}(\omega_s)/R_r$$

Evaluation of $A_{b1}^r(r,\theta,t)$:

Following a similar procedure as for $A_{a1}^r(r,\theta,t)$ and with the same type of approximations for the series terms one obtains

$$A_{b1}^r(r,\theta,t) = \left[\frac{C_0}{\omega_s} \frac{X_1^{rr}(r,\omega_s)\cos\theta \cos(\omega_s t - \psi_r)}{\sqrt{(R_r)^2 + (\mu_0 c X_1^{rr}(\omega_s))^2}} \right.$$

$$- \frac{C_0}{c} \cdot \frac{R_r a \cos\theta [r/b + b/r]}{[(R_r)^2 + (\omega_s L_r)^2][b/a - a/b]} e^{-\frac{R_r}{L_r}t}$$

$$-\sum_{\nu=0}^{\infty}(-1)^{\nu}\ \frac{2C_0\cos\theta}{\mu_0(b-a)\beta_{\nu}^2}\ \sqrt{\frac{a}{r}}\ \cos\left[\frac{b-r}{b-a}\cdot\frac{\pi}{2}(1+2\nu)\right]e^{-d_{\nu}t}\sin\beta_{\nu}t\right] \quad \ldots 9.28$$

Note that

$$X_1^{rr}(r,\omega_s) = \frac{\left[J_1'(\omega_s b/c)\ Y_1(\omega_s r/c) - Y_1'(\omega_s b/c)\ J_1(\omega_s r/c)\right]}{\left[J_1'(\omega_s a/c)\ Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\ J_1'(\omega_s b/c)\right]}$$

$$\doteq\ \frac{\omega_s a}{c}\ \frac{\left[r/b + b/r\right]}{\left[b/a - a/b\right]}\ , \quad |\omega_s| \ll |c|$$

$$\mu_0 c X_1^{rr}(\omega_s) = \mu_0 c\ \frac{\left[J_1'(\omega_s b/c)Y_1(\omega_s a/c) - Y_1'(\omega_s b/c)\ J_1(\omega_s a/c)\right]}{\left[J_1'(\omega_s a/c)\ Y_1'(\omega_s b/c) - Y_1'(\omega_s a/c)\ J_1'(\omega_s b/c)\right]}$$

$$\doteq\ \mu_0\omega_s a\ \frac{\left[b/a + a/b\right]}{\left[b/a - a/b\right]}\ , \quad |\omega_s| \ll |c|$$

$$= \omega_s L^r$$

and the series terms varying as $1/\beta_{\nu}^3$ in equation (9.27) may be neg-lected compared to those varying as $1/\beta_{\nu}^2$ in equation (9.28).

With the above approximations add equation (9.27) and (9.28) to obtain the resultant air gap vector potential due to the rotor current such that

$$A_{t1}(r,\theta,t) = \left[\mu_0 \mathcal{V}_0 \frac{a^2 r}{|z| d} \left(1 + \frac{b^2}{r^2}\right) \cos(\theta - \omega_s t + \psi_r)\right.$$

$$- \mu_0 \mathcal{V}_0 \frac{a^2 r}{|z| d} \left(1 + \frac{b^2}{r^2}\right) \cos(\theta + \psi_r) e^{-\alpha t}$$

$$\left. - \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{2c \mathcal{V}_0}{g \beta_{\nu}^2} \sqrt{\frac{a}{r}} \cos\theta \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right] e^{-\alpha_{\nu} t} \sin\beta_{\nu} t \right]$$

$$\cdots 9.29$$

where

$$d = b^2 - a^2$$

$$g = b - a$$

$$\mathcal{V}_0 = \omega_s L I_1^s$$

$$|z| = \sqrt{\left[(R_r)^2 + (\omega_s L_r)^2\right]}$$

$$\alpha = \frac{R_r}{L_r}$$

The resultant vector potential due to the stator currents only is obtained by adding equations (9.3) and (9.4)

$$A_{t1}^s(r,\theta,t) = \frac{\mu_0 c I_1^s}{\omega_s} \left[\frac{[J_1'(\omega_s a/c) Y_1(\omega_s r/c) - Y_1'(\omega_s a/c) \cdot J_1(\omega_s r/c)]}{[Y_1'(\omega_s b/c) J_1'(\omega_s a/c) - Y_1'(\omega_s a/c) J_1'(\omega_s b/c)]}\right] \cdot$$

$$\left[\sin(\theta - \omega_s t)\right]$$

$$\doteq \mu_0 I_1^s \frac{b^2 r}{d} \left(1 + \frac{a^2}{r^2}\right) \sin(\theta - \omega_s t) \quad \cdots 9.30$$

9.4 Transient Field Intensities

Since $|\dot{a}_\nu| \ll \beta_\nu$, the electric field intensities are

$$E_{t1}^r(r,\theta,t) = - \frac{\partial A_{t1}^r}{\partial t}$$

$$= \left[-\mu_0 v_0 \frac{\omega_s a^2 r}{|z| d} \left(1 + \frac{b^2}{r^2} \right) \sin(\theta - \omega_s t + \psi_r) \right.$$

$$\left. - \mu_0 v_0 \frac{\alpha a^2 r}{|z| d} \left(1 + \frac{b^2}{r^2} \right) \cos(\theta + \psi_r) e^{-\alpha t} \right.$$

$$\left. + \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2c v_0}{g \beta_\nu} \sqrt{\frac{a}{r}} \cos\theta \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu) \right] e^{-\alpha_\nu t} \cos\beta_\nu t \right]$$

$$\dots 9.31$$

and

$$E_{t1}^s(r,\theta,t) = - \frac{\partial A_{t1}^s}{\partial t}$$

$$= \mu_0 I_1^s \frac{\omega_s b^2 r}{d} \left(1 + \frac{a^2}{r^2} \right) \cos(\theta - \omega_s t) \quad \dots 9.32$$

The magnetic field intensities are obtained by using

$$\vec{H} = \vec{1}_r \frac{1}{\mu_0 r}\left(\frac{\partial A}{\partial \theta} \right) + \vec{1}_\theta \frac{1}{\mu_0}\left(-\frac{\partial A}{\partial r} \right)$$

such that

$$\vec{H}_{t1}^{r} = \vec{1}_{r}\left[-\frac{v_0 a^2}{|z|d}\left(1+\frac{b^2}{r^2}\right)\sin(\theta-\omega_s t+\Psi_r)\right.$$

$$+\frac{v_0 a^2}{|z|d}\left(1+\frac{b^2}{r^2}\right)\sin(\theta+\Psi_r)e^{-\alpha t}$$

$$\left.+\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{2c v_0}{\mu_0 g \beta_{\nu}^2}\frac{1}{\nu}\sqrt{\frac{a}{r}}\sin\theta\cos\left[\frac{b-r}{b-a}\cdot\frac{\pi}{2}(1+2\nu)\right]e^{-\alpha_{\nu}t}\sin\beta_{\nu}t\right]$$

$$+\vec{1}_{\theta}\left[-\frac{v_0 a^2}{|z|d}\left(1-\frac{b^2}{r^2}\right)\cos(\theta-\omega_s t+\Psi_r)\right.$$

$$+\frac{v_0 a^2}{|z|d}\left(1-\frac{b^2}{r^2}\right)\cos(\theta+\Psi_r)e^{-\alpha t}$$

$$+\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{2c v_0 \beta_{\nu}'}{\mu_0 g \beta_{\nu}^2}\sqrt{\frac{a}{r}}\cos\theta\sin\left[\frac{b-r}{b-a}\frac{\pi}{2}(1+2\nu)\right]e^{-\alpha_{\nu}t}\sin\beta_{\nu}t$$

$$\left.-\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{c v_0}{\mu_0 g \beta_{\nu}^2}\frac{1}{r}\sqrt{\frac{a}{r}}\cos\theta\cos\left[\frac{b-r}{b-a}\cdot\frac{\pi}{2}(1+2\nu)\right]e^{-\alpha_{\nu}t}\sin\beta_{\nu}t\right]$$

$$\ldots 9.33$$

and

$$\vec{H}_{t1}^{s} = \vec{1}_{r}\left[\frac{I_1^s b^2}{d}\left(1+\frac{a^2}{r^2}\right)\cos(\theta-\omega_s t)\right]$$

$$+\vec{1}_{\theta}\left[-\frac{I_1^s b^2}{d}\left(1-\frac{a^2}{r^2}\right)\sin(\theta-\omega_s t)\right] \qquad \ldots 9.34$$

where

$$\beta_{\nu}' = \frac{\pi}{2}\cdot\frac{(1+2\nu)}{(b-a)}$$

9.5 Transient Torque on the Rotor

From section (6.4) the torque on the rotor is

$$T_{em}^r(t) = \lim_{r \to a} \int_S \vec{r} \times (M.\vec{n})dS = \lim_{r \to a} \int_0^{2\pi} r^2 \, H_\theta H_r \, d\theta$$

$$= \lim_{r \to a} \int_0^{2\pi} r^2 (H_{t1\theta}^s + H_{t1\theta}^r)(H_{t1r}^s + H_{t1r}^r) d\theta$$

It may be noted that at $r = a$, $\cos(b-a/b-a)\frac{\pi}{2}(1+2\nu) = 0$; hence the last term in equation (9.33) will vanish. Also $\sin(b-a/b-a)\frac{\pi}{2}(1+2\nu)$ $= (-1)^\nu$. Furthermore, the magnitude of terms involving product of series will vary at least as $1/\beta_\nu^2$ and will be negligible in comparison with other terms and hence will be neglected. Therefore,

$$T_{em}^r(t) = T_{eml}^m(t) + T_{em2}^r(t) + T_{em3}^r(t) \qquad \ldots 9.35$$

The quantities $T_{eml}^r(t)$, $T_{em2}^r(t)$, and $T_{em3}^r(t)$ are to be defined as follows

$$T_{eml}^r(t) = \lim_{r \to a} \mu_0 r^2 \int_0^{2\pi} \left[-\frac{I_1^s b^2}{d}(1-\frac{a^2}{r^2}) \sin(\theta-\omega_s t) \right.$$

$$\left. - \frac{v_0 a^2}{|Z|d}(1-\frac{b^2}{r^2}) \cos(\theta-\omega_s t+\Psi_r) \right].$$

$$\left[\frac{I_1^s b^2}{d}(1+\frac{a^2}{r^2}) \cos(\theta-\omega_s t) \right.$$

$$\left. - \frac{v_0 a^2}{|Z|d}(1+\frac{b^2}{r^2}) \sin(\theta-\omega_s t+\Psi_r) \right] d\theta$$

$$= \left[\frac{2\mu_0 \pi I^s v_0 a^2 b^2 \cos\Psi_r}{|Z|d} \right] \qquad \ldots 9.36$$

$$T^r_{em2}(t) = \lim_{r \to a} \mu_0 r^2 \int_0^{2\pi} \left[- \frac{I_1^s v_0 a^2 b^2}{|Z| d^2}\left(1 - \frac{a^2}{r^2}\right)\left(1 + \frac{b^2}{r^2}\right) e^{-\alpha t} \sin(\theta - \omega_s t)\sin(\theta + \Psi_r) \right.$$

$$- \frac{v_0^2 a^4}{|Z|^2 d^2}\left(1 - \frac{b^4}{r^4}\right) e^{-\alpha t}\cos(\theta - \omega_s t + \Psi_r)\sin(\theta + \Psi_r)$$

$$+ \frac{I_1^s v_0 a^2 b^2}{|Z| d^2}\left(1 + \frac{a^2}{r^2}\right)\left(1 - \frac{b^2}{r^2}\right) e^{-\alpha t}\cos(\theta - \omega_s t)\cos(\theta + \Psi_r)$$

$$\left. - \frac{v_0^2 a^4}{|Z|^2 d^2}\left(1 - \frac{b^4}{r^4}\right) e^{-\alpha t}\sin(\theta - \omega_s t + \Psi_r)\cos(\theta + \Psi_r) \right] d\theta$$

$$= - \left[\frac{2\mu_0 \pi I_1^s v_0 a^2 b^2}{|Z| d} e^{-\alpha t}\cos(\omega_s t - \Psi_r) \right]$$

$$\ldots 9.37$$

$$T^r_{em3}(t) = \lim_{r \to a} \mu_0 r^2 \int_0^{2\pi} \left[\left[\sum_{\nu=0}^{\infty} \frac{2c v_0 \beta'_\nu}{\mu_0 g \beta_\nu^2}\sqrt{\frac{a}{r}}\cos\theta\, e^{-a_\nu t}\sin\beta_\nu t \right] \cdot \right.$$

$$\left[\frac{I_1^s b^2}{d}\left(1 + \frac{a^2}{r^2}\right)\cos(\theta - \omega_s t) \right.$$

$$- \frac{v_0 a^2}{|Z| d}\left(1 + \frac{b^2}{r^2}\right)\sin(\theta - \omega_s t + \Psi_r)$$

$$\left.\left. + \frac{v_0 a^2}{|Z| d}\left(1 + \frac{b^2}{r^2}\right) e^{-\alpha t}\sin(\theta + \Psi_r) \right] \right] d\theta$$

$$= \left[+ \sum_{\nu=0}^{\infty} \frac{2\pi c I_1^s v_0 a^2 b^2}{g d}\frac{\beta'_\nu}{\beta_\nu^2}\left(1 + \frac{a^2}{a^2}\right)\cos\omega_s t\, e^{-a_\nu t}\sin\beta_\nu t \right.$$

$$+ \sum_{\nu=0}^{\infty} \frac{2\pi c v_0 v_0 a^2 a^2}{|Z| g d}\frac{\beta'_\nu}{\beta_\nu^2}\left(1 + \frac{b^2}{a^2}\right)\sin(\omega_s t - \Psi_r) e^{-\alpha_\nu t}\sin\beta_\nu t$$

$$\left. + \sum_{\nu=0}^{\infty} \frac{2\pi c v_0 v_0 a^2 a^2}{|Z| g d}\frac{\beta'_\nu}{\beta_\nu^2}\left(1 + \frac{b^2}{a^2}\right)\sin\Psi_r\, e^{-(\alpha + \alpha_\nu) t}\sin\beta_\nu t \right]$$

$$\ldots 9.38$$

But

$$\beta'_\nu = \frac{\pi}{2} \frac{(1+2\nu)}{(b-a)} = \frac{\pi}{2g}(1+2\nu)$$

$$\beta_\nu = \frac{\pi}{2}c\frac{(1+2\nu)}{(b-a)} = \frac{\pi c}{2g}(1+2\nu)$$

and

$$L \doteq 2\mu_0 \frac{a^2 b^2}{a} \frac{a^2 b^2}{b^2-a^2} = \frac{2\mu_0}{a} \frac{a^2 b^2}{d}$$

$$\left[\text{See equation (9.11) and put} \quad \left[J'_1\left(\omega_s a/c\right)Y'_1\left(\omega_s b/c\right) - Y'_1\left(\omega_s a/c\right)J'_1\left(\omega_s b/c\right)\right]\right.$$

$$\left. \doteq -\frac{c^2}{\pi\omega_s^2}\cdot\frac{b^2-a^2}{a^2 b^2}\right]$$

Hence, the torque on the rotor

$$T^r_{em}(t) = \frac{2\mu_0\pi I_1^s v_0 a^2 b^2 \cos\psi_r}{|Z| d} - \frac{2\mu_0\pi I_1^s v_0 a^2 b^2}{|Z| d}e^{-\alpha t}\cos(\omega_s t - \psi_r)$$

$$+ \sum_{\nu=0}^{\infty} \frac{4 I_1^s v_0 a^2 b^2}{cd(1+2\nu)}\left(1+\frac{a^2}{a^2}\right)\cos\omega_s t\, e^{-\alpha_\nu t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$+ \sum_{\nu=0}^{\infty} \frac{4 v_0 v_0 a^2 a^2}{c|Z|d(1+2\nu)}\left(1+\frac{b^2}{a^2}\right)\sin(\omega_s t - \psi_r)\, e^{-\alpha_\nu t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$+ \sum_{\nu=0}^{\infty} \frac{4 v_0 v_0 a^2 a^2}{c|Z|d(1+2\nu)}\left(1+\frac{b^2}{a^2}\right)\sin\psi_r\, e^{-(\alpha+\alpha_\nu)t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$\ldots 9.39$$

This is the rate at which electromagnetic angular momentum is flowing

inward through the rotor surface.

9.6 Storage of Electromagnetic Angular Momentum in the Air Gap

From equation (6.12) the electromagnetic angular momentum stored per unit axial length of the air gap is

$$G_\omega = \frac{1}{c^2} \int_a^b \int_0^{2\pi} r^2 S_\theta \, dr \, d\theta$$

$$= \frac{1}{c^2} \int_a^b \int_0^{2\pi} r^2 \left(E_{t1}^S + E_{t1}^r \right) \left(H_{t1r}^S + H_{t1r}^r \right) dr \, d\theta$$

$$= \frac{1}{c^2} \int_a^b \int_0^{2\pi} r^2 \left[\frac{\mu_0 \omega_s I_1^S b^2 r}{d} \left(1 + \frac{a^2}{r^2}\right) \cos(\theta - \omega_s t) \right.$$

$$- \frac{\mu_0 \omega_s v_0 a^2 r}{|Z| d}\left(1 + \frac{b^2}{r^2}\right) \sin(\theta - \omega_s t + \psi_r)$$

$$- \frac{\mu_0 a v_0 a^2 r}{|Z| d}\left(1 + \frac{b^2}{r^2}\right) e^{-at} \cos(\theta + \psi_r)$$

$$+ \sum_{\nu=0}^{\infty} (-)^\nu \frac{2c v_0}{g \beta_\nu} \sqrt{\frac{a}{r}} \cos\theta \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2} (1+2\nu) e^{-a_\nu t} \cos\beta_\nu t \right] \cdot$$

$$\frac{I_1^S b^2}{d}\left(1 + \frac{a^2}{r^2}\right) \cos(\theta - \omega_s t)$$

$$- \frac{v_0 a^2}{|Z| d}\left(1 + \frac{b^2}{r^2}\right) \sin(\theta - \omega_s t + \psi_r)$$

$$\left. + \frac{v_0 a^2}{|Z| d}\left(1 + \frac{b^2}{r^2}\right) e^{-at} \sin(\theta + \psi_r) \right] dr \, d\theta \quad \ldots 9.40$$

Evaluating the integral with respect to θ

$$G_\omega = \frac{1}{c^2} \int_a^b \left[\mu_0 \pi \omega_s I_1^S I_1^S \frac{b^4}{d^2} r^3 \left(1 + \frac{a^2}{r^2}\right)^2 + \mu_0 \pi \omega_s v_0 v_0 \frac{a^4}{|z|^2 d^2} r^3 \left(1 + \frac{b^2}{r^2}\right)^2 \right.$$

$$- 2\mu_0 \pi \omega_s I_1^S v_0 \frac{a^2 b^2}{|z| d^2} r^3 \left(1 + \frac{a^2}{r^2}\right)\left(1 + \frac{b^2}{r^2}\right) \sin\psi_r$$

$$- \mu_0 \pi I_1^S v_0 \frac{a^2 b^2}{d^2 L_r} r^3 \left(1 + \frac{a^2}{r^2}\right)\left(1 + \frac{b^2}{r^2}\right) e^{-\alpha t} \cos(\omega_s t + 2\psi_r)$$

$$- \mu_0 \pi v_0 v_0 \frac{a^2 a^2}{|z| d^2 L_r} r^3 \left(1 + \frac{b^2}{r^2}\right)^2 e^{-\alpha t} \sin(\omega_s t + \psi_r)$$

$$+ \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2\pi c\, I_1^S v_0 b^2}{g\, d\, \beta_\nu} r^2 \sqrt{\frac{a}{r}} \left(1 + \frac{a^2}{r^2}\right) \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right] \cos\omega_s t\, e^{-\alpha_\nu t} \cos\beta_\nu t$$

$$+ \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2\pi c\, v_0 v_0 a^2}{g |z| d\, \beta_\nu} r^2 \sqrt{\frac{a}{r}} \left(1 + \frac{b^2}{r^2}\right) \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right] \sin(\omega_s t - \psi_r) e^{-\alpha_\nu t} \cos\beta_\nu t.$$

$$\left. + \sum_{\nu=0}^{\infty} (-)^\nu \frac{2\pi c\, v_0 v_0 a^2}{g |z| d\, \beta_\nu} r^2 \sqrt{\frac{a}{r}} \left(1 + \frac{b^2}{r^2}\right) \cos\left[\frac{b-r}{b-a} \cdot \frac{\pi}{2}(1+2\nu)\right] \sin\psi_r\, e^{-(\alpha+\alpha_\nu)t} \cos\beta_\nu t. \right] dr$$

$$\dots 9.41$$

Note that

$$\zeta_1^S = \int_a^b r^3 \left(1 + \frac{a^2}{r^2}\right)^2 dr = \frac{b^4}{4} - \frac{5}{4} a^4 + a^2 b^2 + a^4 \ln b/a \qquad \dots 8.75$$

$$\xi_1^r = \int_a^b r^3 \left(1 + \tfrac{b^2}{r^2}\right)^2 dr = \tfrac{5}{4}b^4 - \tfrac{a^4}{4} - a^2 b^2 + b^4 \ln b/a \qquad \text{...8.76}$$

$$\xi_1^{sr} = \int_a^b r^3 \left(1 + \tfrac{a^2}{r^2}\right)\left(1 + \tfrac{b^2}{r^2}\right) dr = \tfrac{3}{4}b^4 - \tfrac{3}{4}a^4 + a^2 b^2 \ln b/a \qquad \text{...8.77}$$

Also it may be shown that

$$\int_a^b \left(r^{3/2} + \tfrac{m}{r^{1/2}}\right) \cos (b-r)\beta_\nu' \, dr$$

$$\doteq \frac{a^{3/2}\left(1 + \tfrac{m}{a^2}\right)(-1)^\gamma}{\beta_\nu'} + \frac{1}{2} \frac{b^{1/2}\left(3 - \tfrac{m}{b^2}\right)}{\beta_\nu'^2}$$

Let

$$\zeta^{sr} = \frac{\sqrt{ab}}{2} \frac{(3 - a^2/b^2)}{\beta_\nu'} \qquad \text{...9.42}$$

$$\zeta^r = \frac{\sqrt{ab}}{2} \frac{(3 - b^2/b^2)}{\beta_\nu'} \qquad \text{...9.43}$$

In terms of the above constants,

$$G_\omega = \frac{1}{c^2}\left[\mu_0 \pi \omega_s I_1^s \frac{b^2 b^2}{d^2} \xi_1^s + \mu_0 \pi \omega_s \upsilon_0 \upsilon_0 \frac{a^2 a^2}{|z|^2 d^2} \xi_1^r - 2\mu_0 \pi \omega_s I_1^s \upsilon_0 \frac{a^2 b^2}{|z| d^2} \zeta_1^{sr} \sin \psi_r \right.$$

$$- \mu_0 \pi I_1^s \upsilon_0 \frac{a^2 b^2}{d^2 L_r} \xi_1^{sr} e^{-\alpha t} \cos(\omega_s t + 2\psi_r)$$

$$- \mu_0 \pi \upsilon_0 \upsilon_0 \frac{a^2 a^2}{|z| d^2 L_r} \xi_1^{sr} e^{-\alpha t} \sin(\omega_s t + \psi_r)$$

$$+ \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{8}{\pi} \frac{I_1^s \, v_0 \, b^2}{(b+a)(1+2\nu)^2} \left[a^2 \left(1 + a^2/a^2\right)(-1)^{\nu} + \zeta^{s\tau} \right] \cos \omega_s t \, e^{-\alpha_\nu t} \cos \frac{\pi}{2} \cdot \frac{(1+2\nu)}{g} ct$$

$$+ \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{8}{\pi} \cdot \frac{v_0 v_0 \, a^2}{|Z|(b+a)(1+2\nu)^2} \left[a^2 \left(1 + b^2/a^2\right)(-1)^{\nu} + \zeta^{\tau} \right] \sin(\omega_s t - \psi_r) e^{-\alpha_\nu t} \cos \frac{\pi}{2} \cdot \frac{(1+2\nu)}{g} ct$$

$$+ \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{8}{\pi} \frac{v_0 v_0 \, a^2}{|Z|(b+a)(1+2\nu)^2} \left[a^2 \left(1 + b^2/a^2\right)(-1)^{\nu} + \zeta^{\tau} \right] \sin \psi_r \, e^{-(\alpha + \alpha_\nu)t} \cos \frac{\pi}{2} \frac{1+2\nu}{g} ct \Biggr]$$

$$\ldots 9.44$$

This is the electromagnetic angular momentum stored in the air gap
as a function of time.

9.7 Stator-Rotor Torque Unbalance and Torque on the Stator.

From equation (6.16), the Stator-Rotor Torque Unbalance is given by

$$\frac{\partial}{\partial t} G_\omega = \text{Rate of change of stored angular momentum in the air gap.}$$

$$\frac{\partial}{\partial t} G_\omega = \frac{1}{c^2} \Biggl[\mu_0 \pi I_1^s v_0 \frac{a^2 b^2 |Z|}{d^2 (L_r)^2} \zeta^{s\tau} e^{-\alpha t} \cos(\omega_s t + \psi_r)$$

$$+ \mu_0 \pi v_0 v_0 \frac{a^2 a^2}{d^2 (L_r)^2} \zeta^{\tau} e^{-\alpha t} \sin \omega_s t$$

$$- \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{4c \, I_1^s v_0 \, b^2}{d(1+2\nu)} \left[a^2 \left(1 + a^2/a^2\right)(-1)^{\nu} + \zeta^{s\tau} \right] \cos \omega_s t \, e^{-\alpha_\nu t} \sin \frac{\pi}{2} \frac{1+2\nu}{g} ct$$

$$-\sum_{\nu=0}^{\infty}(-1)^\nu\,\frac{4c\,V_0 V_0\,a^2}{|Z|\,d\,(1+2\nu)}\left[a^2\left(1+\frac{b^2}{a^2}\right)(-1)^\nu+\varsigma^r\right]\sin(\omega_s t-\psi_r)\,e^{-\alpha_\nu t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$-\sum_{\nu=0}^{\infty}(-1)^\nu\,\frac{4c\,V_0 V_0\,a^2}{|Z|\,d\,(1+2\nu)}\left[a^2\left(1+\frac{b^2}{r^2}\right)(-1)^\nu+\varsigma^r\right]\sin\psi_r\,e^{-(\alpha+\alpha_\nu)t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct\Bigg]$$

$$\ldots 9.45$$

The above equation is true because ω_s and $|\alpha_\nu|$ are much smaller than $\frac{\pi}{2}\cdot\frac{(1+2\nu)}{g}c$.

Torque on the stator

$$T^s_{em}(t)=T^r_{em}(t)+\frac{\partial}{\partial t}G_\omega$$

$$=\left[\frac{2\mu_0\pi\,I_1^s\,V_0\,a^2 b^2\cos\psi_r}{|Z|\,d}-\frac{2\mu_0\pi\,I_1^s\,V_0\,a^2 b^2}{|Z|\,d}\,e^{-\alpha t}\cos(\omega_s t-\psi_r)\right.$$

$$+\frac{\mu_0\pi\,I_1^s\,V_0\,a^2 b^2|Z|}{c^2\,d^2(L_r)^2}\,\varsigma^{sr}_1\,e^{-\alpha t}\cos(\omega_s t+\psi_r)$$

$$+\frac{\mu_0\pi\,V_0\,V_0\,a^2 a^2}{c^2 d^2(L_r)^2}\,\varsigma^r_1\,e^{-\alpha t}\sin\omega_s t$$

$$-\sum_{\nu=0}^{\infty}(-)^\nu\,\frac{4\,I_1^s\,V_0\,b^2\,\varsigma^{sr}}{c\,d\,(1+2\nu)}\cos\omega_s t\,e^{-\alpha_\nu t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$-\sum_{\nu=0}^{\infty}(-1)^\nu\,\frac{4\,V_0\,V_0\,a^2\,\varsigma^r}{c\,|Z|\,d\,(1+2\nu)}\sin(\omega_s t-\psi_r)\,e^{-\alpha_\nu t}\sin\frac{\pi}{2}\frac{(1+2\nu)}{g}ct$$

$$- \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{4 \nu_0 \nu_0 \, a^2 \, s^r}{c|z|d \, (1+2\nu)} \sin \psi_r \, e^{-(q+d_\nu)t} \sin \frac{\pi}{2} \cdot \frac{(1+2\nu)}{g} ct \Bigg] \qquad \ldots 9.46$$

This is the rate at which electromagnetic angular momentum is trans‑
mitted across the air gap surface at r = b (that is stator surface).

9.8 Torque Expressions for small values of the Time.

(A) Consider the rotor torque given by equation (9.39). Define
the Time Constants $\mathcal{T}$ and $\mathcal{T}_\nu$ as

$$\mathcal{T} = \frac{1}{a} \qquad \text{and} \qquad \mathcal{T}_\nu = \frac{1}{d_\nu}$$

In general $\mathcal{T} \gg \mathcal{T}_\nu$ and $\mathcal{T}_\nu$ may be as low as 0.5 second. (Estimates of $\mathcal{T}$ and
$\mathcal{T}_\nu$ will be given later on). Hence, it may be noticed that in
equation (9.39) for small values of t (t may be as high as one tenth of
a millisecond), the first two terms cancel each other and the last
two series cancel each other termwise. Therefore,

$$T^r_{em}(t) = \sum_{\nu=0}^{\infty} \frac{8 I_1^s \nu_0 \, a^2 b^2}{cd \, (1+2\nu)} \cos \omega_s t \, e^{-t/\mathcal{T}_\nu} \sin \frac{\pi}{2} \cdot \frac{(1+2\nu)}{g} ct \, , \quad t \text{ small}$$

$$= \frac{8 I_1^s \nu_0 \, a^2 b^2}{cd} \cos \omega_s t \, e^{-t/\mathcal{T}_\nu} \sum_{\nu=0}^{\infty} \frac{1}{(1+2\nu)} \sin \frac{\pi}{2} \frac{(1+2\nu)}{g} ct \, , \quad t \text{ small}$$

This follows from the fact that $\mathcal{T}_\nu$ is independent of ν .
The above series may be written as

$$S_n = \sum_{n=1,3,\cdots}^{\infty} \frac{\sin n \omega t}{n}$$

where $\qquad \omega = n \frac{c}{2g}$

This is a well known Fourier series whose sum is

$$S_n = \frac{\pi}{4}, \quad 0 < \omega t < \pi$$
$$= -\frac{\pi}{4}, \quad \pi < \omega t < 2\pi$$

It follows that

$$T_{em}^r(t) = \frac{2\pi I_1^s v_0 a^2 b^2}{cd} \cos \omega_s t \, e^{-t/T_v}, \quad 0 < \pi \frac{ct}{2g} < \pi$$

$$= -\frac{2\pi I_1^s v_0 a^2 b^2}{cd} \cos \omega_s t \, e^{-t/T_v}, \quad \pi < \pi \frac{ct}{2g} < 2\pi$$

...9.47

(B) Consider now the stator torque given by equation (9.46). For small values of t due to the same reasoning as for the rotor torque, the first two terms and last two series may be omitted. Furthermore, it may be shown that the magnitudes of the third and the fourth terms are negligible in comparison with the magnitude of any other term. Hence,

$$T_{em}^s(t) = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{4 I_1^s v_0 b^2 g^{s\gamma}}{cd(1+2\nu)} \cos \omega_s t \, e^{-t/T_v} \sin \frac{\pi}{2} \frac{(1+2\nu)}{g} ct, \quad t \text{ small}$$

On further simplification one obtains

$$T_{em}^s(t) = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{4}{\pi} \frac{I_1^s v_0 b^2 g_1^{s\gamma}}{c(b+a)(1+2\nu)^2} \cos \omega_s t \, e^{-t/T_v} \sin \frac{\pi}{2} \frac{(1+2\nu)}{g} ct, \quad t \text{ small}$$

where

$$g_1^{s\gamma} = \sqrt{ab} \left(3 - a^2/b^2 \right)$$

Now the series

$$\sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\sin \frac{\pi}{2} \frac{(1+2\nu)ct}{g}}{(1+2\nu)^2}$$

$$= \sin \omega t - \frac{1}{3^2} \sin 3\omega t + \frac{1}{5^2} \sin 5\omega t - \cdots$$

This is again a Fourier series and may be easily summed by using the 'Geometric series Method'[52]. The sum[53] is

$$
\left.
\begin{array}{l}
S_n = \frac{\pi}{4}\omega t, \quad -\frac{\pi}{2} \leqslant \omega t \leqslant \frac{\pi}{2} \\[2mm]
\quad = \frac{\pi^2}{4} - \frac{\pi}{4}\omega t, \quad \frac{\pi}{2} \leqslant \omega t \leqslant \frac{3\pi}{2}
\end{array}
\right\}
$$

Therefore,

$$
\left.
\begin{array}{l}
T_{em}^s(t) = \frac{\pi}{2} \frac{I_1^s v_0 b^2 g^{sr}_1}{cd} \cos\omega_s t \; e^{-t/J_\nu}_{ct}, \quad -\frac{\pi}{2} \leqslant \pi \frac{ct}{2g} \leqslant \frac{\pi}{2} \\[6mm]
\quad = \frac{\pi}{2} \frac{I_1^s v_0 b^2 g^{sr}_1}{cd} \cos\omega_s t \; e^{-t/J_\nu}(2g-ct), \quad \frac{\pi}{2} \leqslant \pi \frac{ct}{2g} \leqslant \frac{3\pi}{2}
\end{array}
\right\} \text{t small}
$$

$$\cdots 9.48$$

9.9 Torque Expressions for Large Values of the Time

Since $\mathcal{J} \gg \mathcal{J}_\nu$, for $\mathcal{J}_\nu < t < \mathcal{J}$, the contributions of the series terms in equations (9.39) and (9.46) are negligible. Also for reasons stated in the last section, the third and the fourth terms in equation (9.46) may be omitted. Hence,

$$T_{em}^r(t) = \frac{2\mu_o \Pi\, I_1^S\, \mathcal{V}_o a^2 b^2 \cos\psi_r}{|z|d} - \frac{2\mu_o \Pi\, I_1^S\, \mathcal{V}_o a^2 b^2}{|z|d}\, e^{-t/\mathcal{J}}\, \cos(\omega_s t - \psi_r)$$

$$= T_{em}^S(t), \qquad \mathcal{J} > t > \mathcal{J}_\nu \qquad\qquad\qquad ...9.49$$

Also it follows that for $t \gg \mathcal{J}$

$$T_{em}^r = \frac{2\mu_o \Pi\, I_1^S\, \mathcal{V}_o a^2 b^2 \cos\psi_r}{|z|d}$$

$$= T_{em}^S \qquad\qquad\qquad\qquad\qquad ...9.50$$

$$= \text{steady state torque}$$

9.10 Numerical Results

(A) The Time Constants: From equation (9.23)

$$d_\nu = -\frac{c}{2g}\, \ln \frac{[1 - R_r/\mu_o c]}{[1 + R_r/\mu_o c]}$$

The logarithmic function may be expanded[54] as

$$\ln \frac{[1 - R_r/\mu_o c]}{[1 + R_r/\mu_o c]} = -2\left[\frac{R_r}{\mu_o c} + \left(\frac{R_r}{\mu_o c}\right)^2 + \ldots\right], 0 \leq \frac{R_r}{\mu_o c} < 1$$

Since $R_r \ll \mu_o c$, one may write

$$d_v \doteq \frac{R_r}{\mu_o g}$$

Hence,

$$\mathcal{T}_v \doteq \frac{\mu_o g}{R_r} \qquad \qquad \dots 9.51$$

Furthermore, using equation (9.22) one obtains

$$\mathcal{T} = \frac{\mu_o a \, (a^2 + b^2)}{R_r g \, (a+b)} \qquad \qquad \dots 9.52$$

In general, therefore,

$$\mathcal{T} \gg \mathcal{T}_v$$

The time Constants are calculated as functions of the Air gap length for

a = 1 m = Rotor Radius

R_r = 1.72 $\times 10^{-4}$ = Rotor surface resistivity

The results are shown as graphs in Fig. (9.2).

(B) Magnitude of the Torque components: Let equations (9.47), (9.48), and (9.50) be written as

$$
\begin{aligned}
T_{em}^r(t) &= T_1 \, I_1^s I_1^s \cos\omega_s t \, e^{-t/\mathcal{T}_v} , & 0 < \pi \frac{ct}{2g} < \pi \\
&= - T_1 \, I_1^s I_1^s \cos\omega_s t \, e^{-t/\mathcal{T}_v} , & \pi < \pi \frac{ct}{2g} < 2\pi
\end{aligned}
\left. \begin{aligned} \\ \\ \end{aligned} \right\} \text{t small} \quad \dots 9.53
$$

$$
\begin{aligned}
T_{em}^s(t) &= T_2 \, I_1^s I_1^s \cos\omega_s t \, e^{-t/\mathcal{T}_v} ct , & -\frac{\pi}{2} \le \pi \frac{ct}{2g} \le \frac{\pi}{2} \\
&= T_2 \, I_1^s I_1^s \cos\omega_s t \, e^{-t/\mathcal{T}_v} (2g - ct), & \frac{\pi}{2} \le \pi \frac{ct}{2g} \le \frac{3\pi}{2}
\end{aligned}
\left. \begin{aligned} \\ \\ \end{aligned} \right\} \text{t small} \quad \dots 9.54
$$

Since $R_2 \ll \omega_c$, one may write

$$d_0 \doteq \frac{R_2^2}{\omega_c^2}$$

Hence,

$$L_0^2 \doteq \frac{R_2^2}{\omega_c^2} \qquad \text{...9.51}$$

Furthermore, using equation (9.22) one obtains

$$\tau = \frac{\omega_a(z_1^2 + R_2^2)}{R_2^3(z+b)} \qquad \text{...9.52}$$

In general, therefore,

$$\tau \gg \tau_0'$$

The time Constants are calculated as functions of the Air gap length for

$a = 1$ in = Rotor Radius

$R_2 = 1.72 \times 10^{-4}$ = Rotor surface resistivity

The results are shown as graphs in Fig. (32).

(B) Magnitude of the Torque components: Let equations (9.47),

(9.49), and (9.50) be written as

$$T_{em}^{m}(t) = T_1 H_1^2 \cos\omega_1^2 t \, e^{-t/\tau_0} \,, \quad 0 < \tau < \frac{\omega_c}{\omega_a} \tau_0 \quad \left\{ \begin{array}{l} \tau \text{ small} \\ \end{array} \right.$$
$$= -T_1 T_2^2 H_1^2 \cos\omega_1^2 t \, e^{-t/\tau_0} \,, \quad \frac{\omega_c}{\omega_a} \tau_0 < \tau < \tau_0 \qquad \text{...9.53}$$

$$T_{em}^{s}(t) = T_1 \tau_2^2 H_1^2 \cos\omega_1^2 t \, e^{-t/\tau_0} \,, \quad -\frac{\omega_c}{\omega_a} < \frac{\omega_c}{\omega_a} \tau_0 \quad \left\{ \begin{array}{l} \tau \text{ small} \\ \end{array} \right.$$
$$= T_1 H_1^2 \cos\omega_1^2 t \, e^{-t/\tau_0}(z_1^2 - c_1^2) \,, \quad \frac{\omega_c}{\omega_a} \tau_0 < \tau < \frac{\omega_c}{\omega_a} \tau_0 \qquad \text{...9.54}$$

$$T_{em}^r = T_3 I_1^s I_1^s \qquad \qquad \ldots 9.55$$

Where

$$T_1 = 2\pi\omega_s La^2 b^2/cd \qquad \qquad \ldots 9.56$$

$$T_2 = \pi\omega_s Lb^2 \varsigma_1^{sr}/2cd \qquad \qquad \ldots 9.57$$

$$T_3 = 2\mu_c r\omega_s La^2 b^2 \cos\psi_r/|z| d \qquad \qquad \ldots 9.58$$

For the values of a and R_r given in part (A) of this section, the magnitudes of T_1, T_2 and T_3 (these are the torque magnitudes normalized with respect to I_1^s) are calculated as functions of the air gap length. The results are plotted in Fig. (9.3). Note that T_1 gives the normalized value of the rotor torque at $t = 0^+$.

(C) Normalized Torque Versus Time (t small): For the above values of a and R_r, the periodic variation $T_{em}^r(t)$ and $T_{em}^s(t)$ given by equations (9.53) and (9.54) respectively are shown in normalized form in Fig. (9.4). These curves may be used to determine the differential torque impulse on the stator and rotor (i.e. the difference in torque integrated over the time of the transients).

9.11 Discussion

(A) Discussion on the Solution: To advance the concept of electro-magnetic angular momentum flow in an induction motor a transient problem has been investigated. The solution of the field equations is obtained in the exact sense. In the equations giving the electromag-

g - Air Gap Length in m.
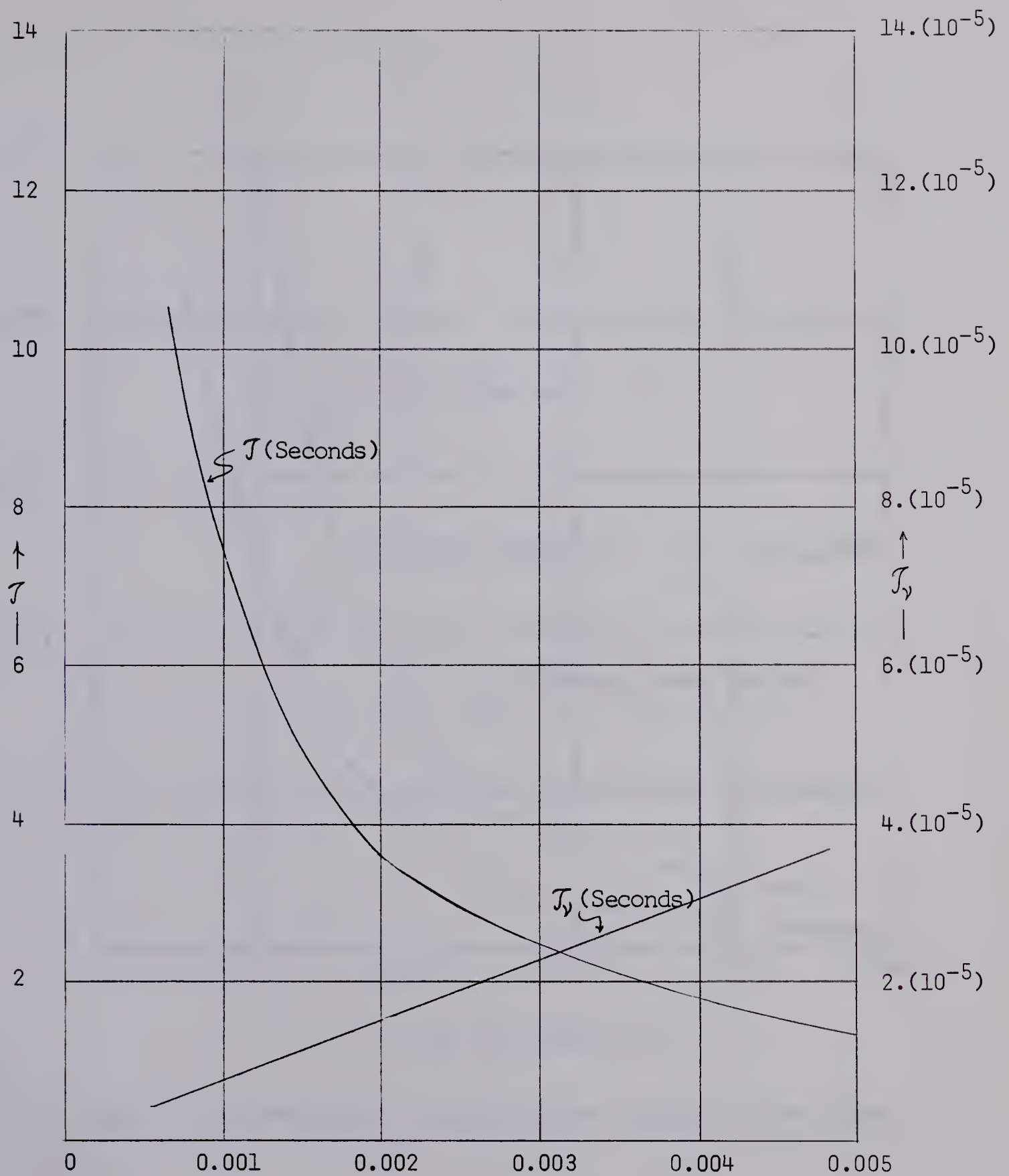
Fig. 9.2 - Time Constants Versus Air Gap Length

Fig. 9.3 - Normalized Torque Magnitudes Versus Air Gap Length
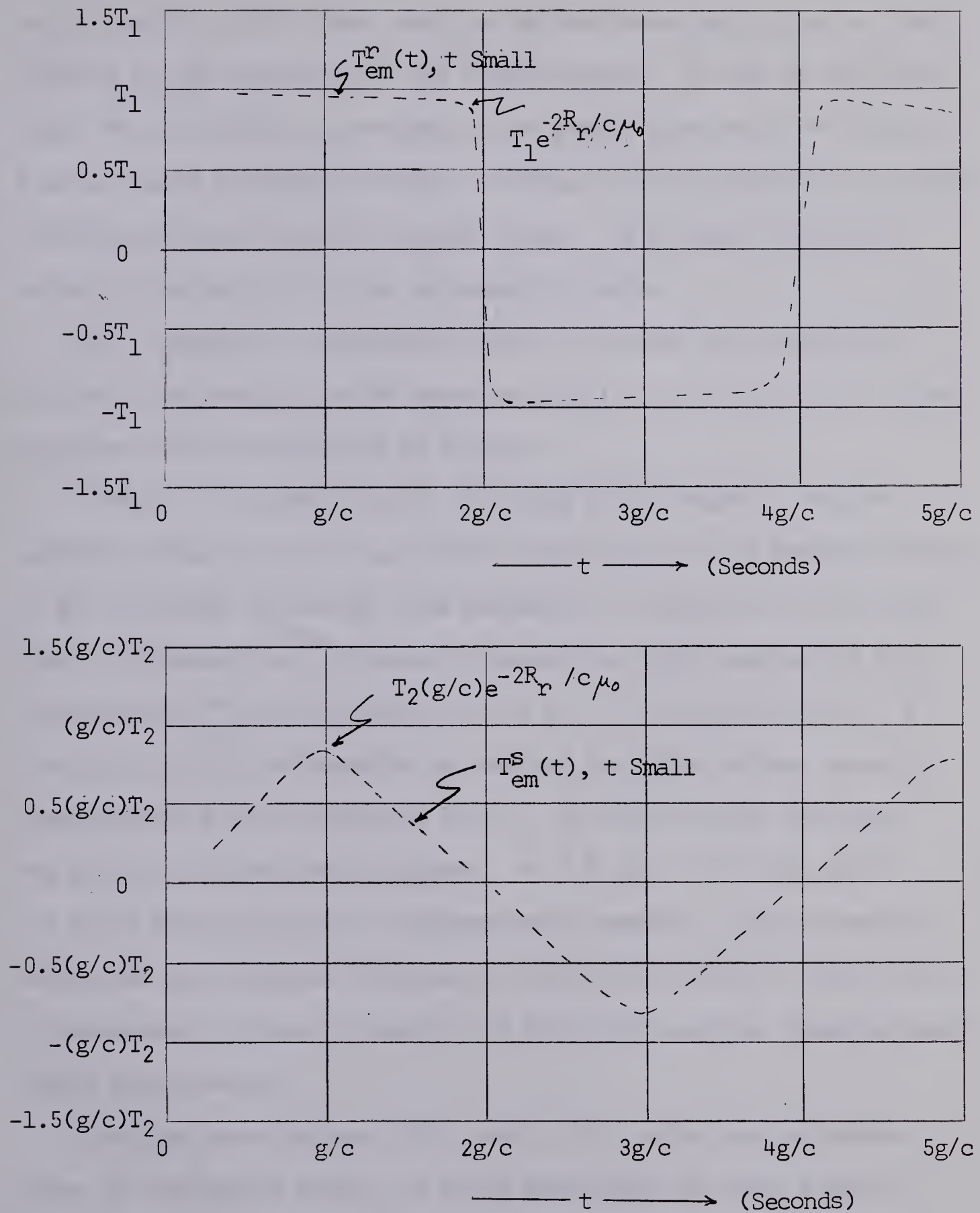
Fig. 9.4 - Normalized Torque Versus Time (t Small)

netic quantities only those terms are omitted whose magnitudes are small compared to the magnitudes of the terms retained. It may be shown that under the quasi-static approximation the series terms will be missing from all these equations and such a solution will not satisfy the initial conditions of rotor vector potential waves. As a result the initial values of the torque will not be properly treated.

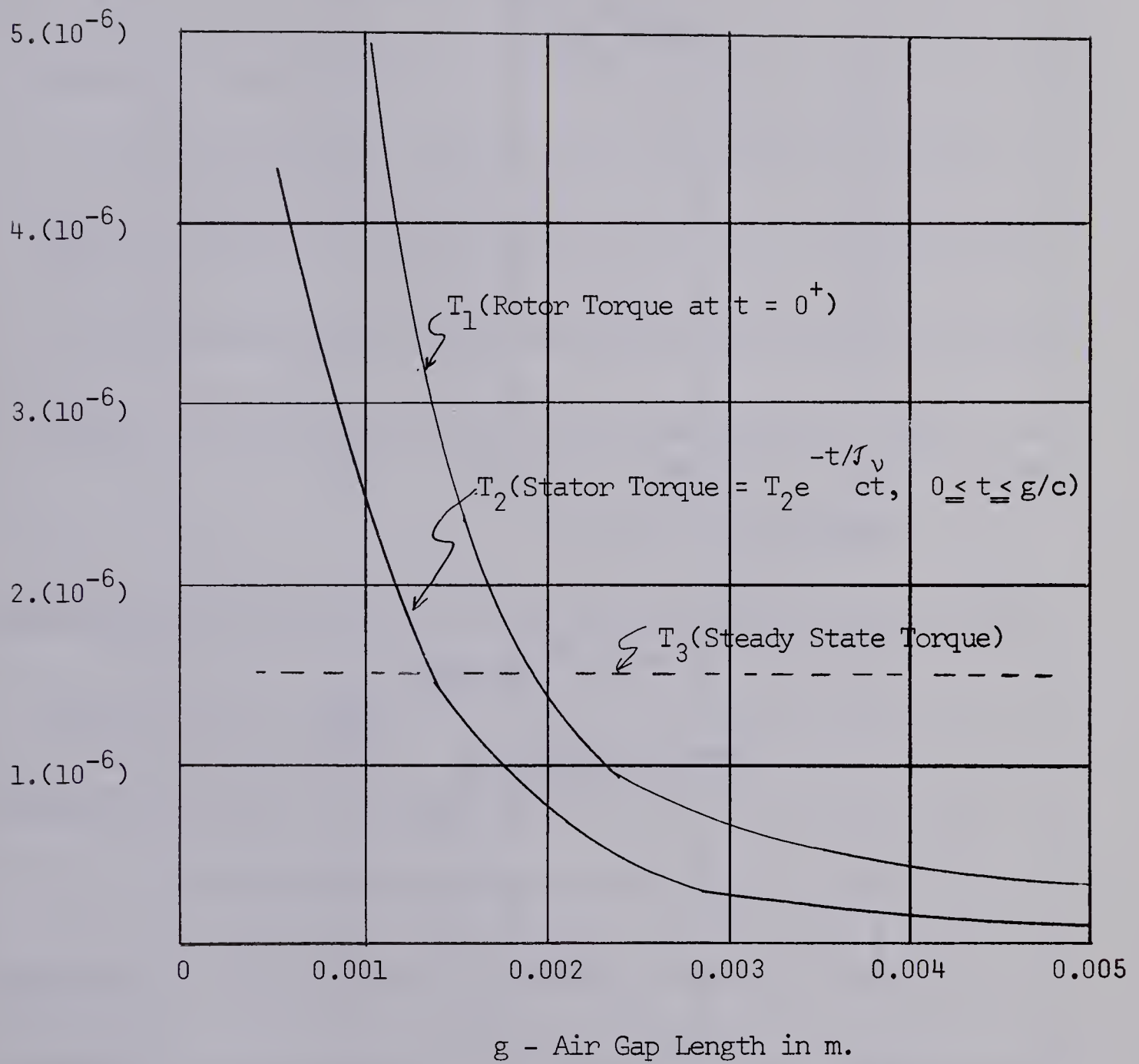(B) Discussion on Momentum Effects: Consider the expressions for the rotor torque given by equation (9.39), (9.47) and (9.49). These equations may be interpreted as follows:

From $t = 0$ the rotor starts absorbing electromagnetic angular momentum stored in the air gap by the stator field. The combined effects of all the terms (including those neglected) in equation (9.39) should give a continuous rate of change of absorption (this removes the discontinuity of $T_{em}^r(t)$ in equation (9.47) at $t = 0$) such that at $t = 0^+$ the rate at which the momentum is crossing the rotor surface inward is approximately given by equation (9.47). As this function decreases the effect of other terms increases. At $t \doteq 2g/c + 0^{++}$ (Fig.(9.4)) the rotor starts giving out electromagnetic momentum. The process of absorption and rejection continues in cyclic order and for $t > T_0$ the rate is approximately given by equation (9.39), which would be obtained under static approximation.

Consider now equations (9.46) and (9.48), which are the expressions for the stator torque. A close examination of these equations using typical numerical values will show that for very small values of t (of the order of fraction of a $\mu \mu s$) the stator torque is approximately zero. This is an indication that the torque on the stator comes

from a delayed action, which is in support of the momentum concept. At $t > \mathcal{T}_y$ the stator and rotor torques are equal in magnitude and are given by equation (9.49).

From the above discussions it follows that the quasi-static approach can not give proper account of the torque at the first instants ($t < \mathcal{T}_y$) during the course of the transient. The time functions associated with the large roots of the wave system are responsible for the initial flow of the momentum.

(c) Discussion on the Magnitude of the Torque Components: The minimum air gap length[55] of an induction machine with a = 1 m is about .0017m. From Fig. (9.3) it follows that T_1/T_3 giving the transient torque at t = 0^+ to the steady state torque for R_r = 1.72 (10^{-4}) is about 1.3. This ratio increases if R_r decreases (T_1 constant, T_3 decreases with R_r almost in direct proportion). In a real machine normally R_r would be larger and T_1/T_3 would be smaller than the above value.

CHAPTER 10

CONCLUSIONS

The law of conservation of momentum has been interpreted for an electromagnetic system. This has led to a concept of a conversion of electromagnetic angular momentum to mechanical angular mometum to produce torque. With a view to apply this in the theory of electrical machines, an expression has been obtained in terms of Maxwell's stress tensor and electromagnetic angular momentum density to calculate the electromagnetic torque on a body in free space.

The electromagnetic field problem of double cylindrical machines has been reduced to finding the transient and the steady state solutions of a vector wave equation. A general solution of this equation has been developed by using the method of the Laplace transform. This exact solution may be utilized for the harmonic (space and time) analysis of certain types of double cylindrical machines.

The machine volt-ampere equations have been obtained by integrating the electric field intensities and the torque equation has been deduced from the momentum concept. The quasi-static field quantities and the resulting machine equations of motion have also been derived from the low argument approximations of the Bessel functions involved in the exact equations. These results are in agreement with those found by other methods.

A detailed wave theory analysis of an induction machine has been developed in the exact sense to study the production of torque in terms of a transfer of electromagnetic angular momentum across the air gap. It has been shown that the quasi-static approximation gives

completely satisfactory quantitative results (accuracy - one part in about 10^{10}) in the case of the machine under steady state conditions. However, in the transient case such an approximation cannot provide information about the torque for the first instants (of the order of μ seconds) after the transient has started. An exact solution is necessary, and in such a solution the time functions associated with the large roots of the electromagnetic wave system, obtained in the form of infinite series, are responsible for the initial flow of electromagnetic angular momentum.

No doubt, one is fully entitled to explain the production of torque in the induction machine from the concept of a flow of electromagnetic angular momentum and its conversion into mechanical angular momentum.

APPENDIX 1

In this appendix it will be shown that

$$\epsilon_0\left[\vec{E}\,(\nabla.\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]$$

$$+\frac{1}{\mu_0}\left[\vec{B}(\nabla.\vec{B}) - \vec{B}\times(\nabla\times\vec{B})\right]$$

$$= M_{\alpha\beta}\cdot\overleftrightarrow{\nabla} \triangleq div\,M \triangleq i_\alpha \sum_{\beta=1}^{3}\frac{\partial M_{\alpha\beta}}{\partial\beta}, \quad \alpha=1,2,3 \qquad \ldots A1.1$$

where $\quad M_{\alpha\beta}= \epsilon_0\vec{E}\vec{E} + \frac{1}{\mu_0}\vec{B}\vec{B} - \frac{1}{2}(\epsilon_0\vec{E}.\vec{E} + \frac{1}{\mu_0}\vec{B}.\vec{B})\,U \qquad \ldots A1.2$

Proof:

It is known that

$$\epsilon_0\left[\vec{E}(\nabla.\vec{E}) - E\times(\nabla\times\vec{E})\right]$$

$$= \epsilon_0\left[\left[\vec{E}(\nabla.\vec{E}) +(\vec{E}\,\overleftarrow{\nabla}).\vec{E}\right] - \frac{1}{2}\left[\vec{E}.(\vec{E}\,\overleftarrow{\nabla}) + \vec{E}.(\vec{E}\,\overleftarrow{\nabla})\right]\right]$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ldots A1.3$

$$\frac{1}{\mu_0}\left[\vec{B}(\nabla.\vec{B}) - \vec{B}\times(\nabla\times\vec{B})\right]$$

$$= \frac{1}{\mu_0}\left[\left[\vec{B}(\nabla.\vec{B}) +(\vec{B}\,\overleftarrow{\nabla}).\vec{B}\right] - \frac{1}{2}\left[\vec{B}.(\vec{B}\,\overleftarrow{\nabla}) + \vec{B}.(\vec{B}\,\overleftarrow{\nabla})\right]\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots A1.4$$

Also it can be shown that

$$(\vec{u}.\vec{v})\,\overleftrightarrow{\nabla} = \vec{u}.(\vec{v}\,\overleftrightarrow{\nabla}) + \vec{v}.(\vec{u}\,\overleftrightarrow{\nabla}) \qquad \ldots A1.5$$

$$(\vec{u}\,\vec{v}).\overleftrightarrow{\nabla} = \vec{u}\,(\vec{v}.\overleftrightarrow{\nabla}) + (\vec{u}\,\overleftrightarrow{\nabla}).\vec{v} \qquad \ldots A1.6$$

Furthermore

$$[(\vec{u} \cdot \vec{v})\, U] \cdot \overleftarrow{\nabla} = \nabla (\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \vec{v})\overleftarrow{\nabla} \qquad \ldots A1.7$$

Using the facts given by the last three equations, if equations (A1.3) and (A1.4) are added, the desired result will follow.

APPENDIX 2

In this appendix Maxwell's equations will be solved in terms of suitably defined potential functions.

Since $\nabla \cdot \vec{B} = 0$, one may write

$$\vec{B} = \nabla \times \vec{A} \qquad \qquad \text{... A2.1}$$

From $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ and equation (A2.1) it follows that

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \qquad \qquad \text{... A2.2}$$

where $\vec{A}$ and Φ and electromagnetic vector and scalar potentials respectively.

Substituting equation (A2.1) and (A2.2) in

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{D} = P$$

and using the fact that for homogeneous and isotropic medium stationary with respect to the reference system $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$, one obtains

$$\nabla \times \nabla \times \vec{A} + \mu \epsilon \nabla \frac{\partial \Phi}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = \mu \vec{J}_f \qquad \qquad \text{... A2.3}$$

$$\nabla^2 \Phi + \nabla \cdot \frac{\partial \overline{A}}{\partial t} = -P/\epsilon \qquad \qquad \text{... A2.4}$$

For identical boundary conditions all particular solutions of (A2.3) and (A2.4) would lead to the same electromagnetic fields. Since $\vec{A}$ and Φ are not uniquely determined by (A2.1) and (A2.2.), a further restriction may be applied in order to simplify the equations. One such

restriction is

$$\nabla \cdot \vec{A} + \mu \epsilon \frac{\partial \Phi}{\partial t} = 0 \qquad \ldots \text{A2.5}$$

which is known as the Lorentz condition.

Substitution of equation (A2.5) in equations (A2.3) and (A2.4) yields

$$\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = - \mu \vec{J}_f \qquad \ldots \text{A2.6}$$

$$\nabla^2 \Phi - \mu \epsilon \frac{\partial^2 \Phi}{\partial t^2} = - \rho / \epsilon \qquad \ldots \text{A2.7}$$

For the class of problems dealt with in this thesis, $\rho = 0$ in all free space regions and, then it is possible to choose $\Phi = 0$. This is equavelent to the condition

$$\nabla \cdot \vec{A} = 0 \qquad \ldots \text{A2.8}$$

APPENDIX 3

The evaluation of the residues in chapter 9 involves the use of the roots of the function

$$R_r + Q \frac{\left[K'_1(kb) I_1(ka) - I'_1(kb)K_1(ka)\right]}{\left[K'_1(ka) I'_1(kb) - I'_1(ka)K'_1(kb)\right]} = 0$$

... A3.1

where

$$R_r = \text{Rotor surface resistivity}, \quad k = \frac{s}{c}$$

and,

$$Q = \mu_o c = \text{Constant such that} \quad Q \gg R_r$$

The required roots will be determined as follows:

(a) Large Roots: Normally the large roots[50] of the Bessel functions and certain other related functions are obtained by using the asymptotic expansions of these functions valid for large argu-ments in the required sector of the complex plane. This principle and the properties[51] of asymptotic series may be applied to find the large roots of equation (A3.1). The modified Bessel functions of order one have the asymptotic expansion[50],

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} U(z) + \frac{e^{-z+j\frac{3\pi}{2}}}{\sqrt{2\pi z}} V(z)$$

$$-\frac{\pi}{2} < \text{angle } z < \frac{3\pi}{2}$$

... A3.2

and

$$K_1(z) = \sqrt{\frac{\pi}{2z}} \ e^{-z}V(z), \quad |\text{angle } z| < \frac{3\pi}{2}$$

... A3.3

where,

$$U(z) = 1 - \frac{A_1}{z} + \frac{A_2}{z^2} - \frac{A_3}{z^3} + \frac{A_4}{z^4} - \frac{A_5}{z^5} + \ldots$$

$$\ldots \text{ A3.4}$$

$$V(z) = 1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \frac{A_4}{z^4} + \frac{A_5}{z^5} + \ldots$$

$$\ldots \text{ A3.5}$$

where,

$$A_1 = \frac{3}{1!\,8}, \quad A_2 = -\frac{15}{2!\,8^2}, \quad A_3 = \frac{315}{3!\,8^3},$$

$$A_4 = -\frac{14175}{4!\,8^4}, \quad A_5 = \frac{14175.(77)}{5!\,8^5}$$

Let

$$U(z) = e^{-\theta_1}$$

$$V(z) = e^{+\theta_1}$$

Hence

$$I_1(z) = \frac{e^{z-\theta_1}}{\sqrt{2\pi z}} + \frac{e^{-z+j\frac{3\pi}{2} + \theta_1}}{\sqrt{2\pi z}} \qquad \ldots \text{ A3.6}$$

Differentiating equation (A3.6), one obtains

$$I'_1(z) = \frac{1}{\sqrt{2\pi z}}\left[e^{z-\theta'_1} - e^{-z+j\frac{3\pi}{2} + \theta'_1}\right]$$

$$\ldots \text{ A3.7}$$

where,

$$e^{-\theta'_1} = 1 - \frac{B_1}{z} + \frac{B_2}{z^2} - \frac{B_3}{z^3} + \frac{B_4}{z^4} - \frac{B_5}{z^5} + \ldots$$

$$\ldots \text{ A3.8}$$

$$e^{+\theta'_1} = 1 + \frac{B_1}{z} + \frac{B_2}{z^2} + \frac{B_3}{z^3} + \frac{B_4}{z^4} + \frac{B_5}{z^5} + \ldots$$

$$\ldots \text{ A3.9}$$

with

$$B_1 = A_1 + \frac{1}{2}$$

$$B_n = A_n + \frac{(2n-1)}{2} A_{n-1}, \qquad n \geq 2$$

Similarly

$$K_1(z) = \sqrt{\frac{\pi}{2z}} \; e^{-z+\theta_1} \qquad \qquad \ldots \text{ A3.10}$$

and

$$K'_1(z) = -\sqrt{\frac{\pi}{2z}} \; e^{-z+\theta'_1} \qquad \qquad \ldots \text{ A3.11}$$

Let

$$\frac{b}{a} = \rho, \; ka = z$$

$$\therefore \; kb = \rho z$$

Using these relations with equations (A3.6), (A3.7), (A3.10) and (A3.11), it may be shown that

$$K'_1(kb)I_1(ka) - I'_1(kb)K_1(ka)$$

$$= -\frac{1}{2\sqrt{\rho z}} \cosh\left[(\rho-1)z + \theta_1 - \theta'_1(\rho)\right] \qquad \ldots \text{ A3.12}$$

$$K'_1(ka)I'_1(kb) - I'_1(ka)K'_1(kb)$$

$$= -\frac{1}{2\sqrt{\rho z}} \sinh\left[(\rho-1)z + \theta'_1 - \theta'_1(\rho)\right] \qquad \ldots \text{ A3.13}$$

where, $e^{-\theta'_1(\rho)}$ and $e^{+\theta'_1(\rho)}$ are given by equation (A3.8) and (A3.9) respectively with z replaced by ρz.

Substitution of (A3.12) and (A3.13) into (A3.1) reduces it to

$$R + \frac{\cosh\left[(P-1)z + \theta_1 - \theta'_1 (P)\right]}{\sinh\left[(P-1)z + \theta'_1 - \theta'_1 (P)\right]} = 0 \quad \dots \text{A3.14}$$

where $\quad R = R_r/Q \qquad \dots \text{A3.15}$

Let $\quad (P-1)z = mz$

and $\quad \psi = \theta_1 - \theta'_1 (P)$

$\quad \Phi = \theta'_1 - \theta'_1 (P)$

Hence,

$$\frac{\cosh(mz+\psi)}{\sinh(mz+\Phi)} = -R$$

from which

$$e^{2mz} = \frac{Re^{-\Phi} - e^{-\psi}}{Re^{\Phi} + e^{\psi}} \qquad \dots \text{A3.16}$$

Now

$$Re^{-\Phi} = R\left(1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \frac{C_3}{z^3} + \dots\right)$$

$$Re^{\Phi} = \left(1 - \frac{C_1}{z} + \frac{C_2}{z^2} + \frac{C_3}{z^3} + \dots\right)$$

$$e^{-\psi} = \left(1 + \frac{D_1}{z} + \frac{D_2}{z^2} + \frac{D_3}{z^3} + \dots\right)$$

$$e^{\psi} = \left(1 - \frac{D_1}{z} + \frac{D_2}{z^2} - \frac{D_3}{z^3} + \dots\right)$$

where

$$C_1 = \frac{B_1}{P} - B_1 \qquad \dots \text{A3.17}$$

$$C_n = \frac{B_n}{\rho^n} - \frac{B_{n-1}}{\rho^{n-1}} B_1 + \frac{B_{n-2}}{\rho^{n-2}} B_2 + \ldots + (-1)^n B_n$$

... A3.18

$$D_1 = \frac{B_1}{\rho} - A_1$$

... A3.19

$$D_n = \frac{B_n}{\rho^n} - \frac{B_{n-1}}{\rho^{n-1}} A_1 + \frac{B_{n-2}}{\rho^{n-2}} A_2 + \ldots + (-1)^n A_n$$

... A3.20

Substitution of these relations in equation (A3.16) leads to

$$e^{2mz} = \frac{a_0 + \dfrac{a_1}{z} + \dfrac{a_2}{z^2} + \ldots + \dfrac{a_n}{z^n} + \ldots}{b_0 + \dfrac{b_1}{z} + \dfrac{b_2}{z^2} + \ldots + \dfrac{b_n}{z^n} + \ldots}$$

... A3.21

where

$$a_0 = R - 1 \qquad\qquad b_0 = R + 1$$

$$a_n = RC_n - D_n \qquad\qquad b_n = (-1)^n (RC_1 + D_1)$$

Equation (A3.21) may be expressed as

$$e^{2mz} = F_0 + \frac{F_1}{z} + \frac{F_2}{z^2} + \frac{F_3}{z^3} + \ldots + \frac{F_n}{z^n} + \ldots$$

... A3.22

where F_n may be calculated from

$$a_n = \sum_{r=0}^{n} F_r b_{n-r}$$

... A3.23

Such that

$$F_o = \frac{a_o}{b_o} = \frac{R - 1}{R + 1}$$

$$F_1 = \frac{a_1}{b_0} - \frac{b_1 C_o}{b_o} = \frac{2(R^2 C_1 - D_1)}{(R + 1)^2}$$

$$F_n = \frac{a_n}{b_o} - \frac{1}{b_o} \sum_{r=o}^{n-1} F_r b_{n-1-r}$$

Solution of Equation (A3.22)

Approximate solution of equation (A3.22) will now be obtained by numerical method.

First order approximation:

Let $e^{2mz} = F_o$

The quantity $F_o = \frac{R - 1}{R + 1}$ is a negative number with **magnitude being** slightly less than 1. Hence

$$e^{2mz} = r \langle \Pi \qquad \text{where} \quad r = \frac{1 - R}{1 + R}$$

Solving for z, one obtains

$$\text{Re}(z) = \frac{1}{2m} \ln r \qquad \qquad \text{... A3.24}$$

$$\left. \begin{array}{l} \text{Im}(z) = \frac{\Pi}{2m} (1 \pm 2\gamma) = \beta \quad (\text{say}) \\ \\ \gamma = o,1,2, \ \ldots \end{array} \right\} \qquad \text{... A3.25}$$

Now

$$|\ln r| \dot{=} o, \text{ hence}$$

$$z \dot{=} j\beta$$

Second order approximation

$$e^{2mz} = F_o + \frac{F_1}{j\beta} = F_o - j \frac{F_1}{\beta}$$

$$Re(z) = \frac{1}{4m} \ln \left[F_o^2 + (\frac{F_1}{\beta})^2 \right] \qquad \dots \text{A3.26}$$

$$Im(z) = \beta + \frac{F_1}{2mF_o\beta} \qquad \dots \text{A3.27}$$

Still

$$\dot{z} = j \left(\beta + \frac{F_1}{2mF_o\beta} \right)$$

Third Order Approximation

$$e^{2mz} = F_o - \frac{jF_1}{\beta + \frac{F_1}{2mF_o\beta}} - \frac{F_2}{(\beta + \frac{F_1}{2mF_o\beta})^2}$$

From which

$$Re(z) = \frac{1}{4m} \ln \left[(F_o - \frac{F_2}{\beta^2})^2 + (\frac{F_1}{\beta})^2 \right] \qquad \dots \text{A3.28}$$

$$Im(z) = \beta + \frac{F_1}{2mF_o\beta} \qquad \dots \text{A3.29}$$

Proceeding in the same way it may be obtained that

$$Re(z) = \frac{1}{4m} \ln \left[(M_o + \frac{M_2}{\beta^2} + \frac{M_4}{\beta^4} + \dots)^2 \right.$$

$$\left. + (\frac{M_1}{\beta} + \frac{M_3}{\beta^3} + \frac{M_5}{\beta^5} + \dots)^2 \right] \qquad \dots \text{A3.30}$$

$$Im(z) = \beta + \frac{N_1}{\beta} + \frac{N_3}{\beta^3} + \frac{N_5}{\beta^5} \qquad \dots \text{A 3.31}$$

where

$$M_o = F_o, \quad M_2 = -F_2, \quad M_4 = (2F_2 A + F_4)$$

$$M_1 = F_1, \quad M_3 = - (F_1 A + F_3)$$

$$M_5 = (F_1 A^2 + F_1 B_1 + 3F_3 A + F_5)$$

$$N_1 = \frac{F_1}{2mF_o}, \quad N_3 = - \left(- \frac{F_1 F_2}{2mF_o^2} + \frac{F_1 A + F_3}{2mF_o^4}\right)$$

$$N_5 = \left[\frac{F_1 A^2 + F_1 B_1 + 3F_3 A + F_5}{2mF_o^5} - \frac{F_1(2F_2 A + F_4)}{2mF_o^2}\right.$$

$$\left. + \frac{F_2^2 F_1}{2mF_o^3} - \frac{3F_2(F_1 A + F_3)}{2mF_o^4}\right]$$

$$A = \frac{F_1}{2mF_o}, \quad B_1 = - \left[\frac{F_3 + F_1 A}{2mF_o^3} - \frac{F_1 F_2}{2mF_o^2}\right]$$

$$\dots \text{A 3.32}$$

Equations (A3.30) and (A3.31) representing the real and imaginary parts of z will be sufficiently convergent to give the large roots quite accurately. The earlier roots will be approximate.

Since $ka = \frac{s}{c}a = z$, the large roots of equation (A3.1) are, therefore, given by

$$s = \frac{c}{a} \text{ Re}(z) + \frac{c}{a} \text{ Im}(z) \qquad \dots \text{A3.33}$$

As a first order approximation

$$s = \frac{c}{2(b-a)} \ln \frac{\left[1 - R_n/\mu_o c\right]}{\left[1 + R_n/\mu_o c\right]} \pm j \frac{\pi c}{2} \frac{(1 + 2\nu)}{(b-a)} \quad \nu = 0,1,2\dots$$

$$\dots \text{A 3.34}$$

These roots are in the left half of the complex plane, occuring

in conjugate pairs.

(B) Samll Roots: The small roots of equation (A3.1) may be obtained by using the ascending series for the Bessel functions involved. These series[48] are

$$I_1(z) = \frac{z}{2} + \frac{z^3}{16} + \ldots = \sum_{r=o}^{\infty} \frac{(\frac{1}{2}z)^{1+2r}}{r! \; \tau(r+2)} \qquad \ldots \text{A3.35}$$

$$K_1(z) = \frac{1}{z} + \frac{1}{2}z \ln z + \ldots = \lim_{n \to 1} \frac{I_{-n}(z) - I_n(z)}{\sin n\pi} \qquad \ldots \text{A3.36}$$

Using only the 1st term of each of the above series as in section (7.2), it may be shown that

$$\lim_{s \to 0} \frac{\left[K'_1(kb)I_1(ka) - I'_1(kb)K_1(ka)\right]}{\left[K'_1(ka)I'_1(kb) - I'_1(ka)K'_1(kb)\right]} = \frac{sa}{c} \frac{[b/a + a/b]}{[b/a - a/b]} \qquad \ldots \text{A3.37}$$

Let

$$L_r = \mu_o a \cdot \frac{[b/a + a/b]}{[b/a - a/b]} \qquad \ldots \text{A3.38}$$

Therefore, the smallest zero of the left hand side of (A3.1) is

$$s = - R_r/L_r \qquad \ldots \text{A3.39}$$

Using more terms of the series in equations (A3.35) and (A3.36) estimation shows that for $\frac{b}{a} = \rho$, $o < \rho \leq 1$, the absolute magnitude of the next higher zero of equation (A3.1) is greater than c (magnitude of the velocity of light). If s_2 is this zero,

$$\left|s_2\right| > \left|c\right|$$

$$\left|k\right| > \left|\frac{s_2}{c}\right| > 1$$

Since for this condition the asymptotic series shown in equations (A3.2) through (A3.5) are valid, the numerical value of s_2 may be calculated as in section (A) of this appendix, and is approximately equal to

$$s_2 = \frac{c}{2(b-a)} \ln \frac{\left[1 - R_r/\mu_o c\right]}{\left[1 + R_r/\mu_o c\right]} + j \frac{\pi c}{2(b-a)} \qquad \ldots \text{A3.40}$$

with another zero occurring in conjugate pair.

This completes the determination of all the roots of equation (A3.1), with their first order approximate values given by equations (A3.34) and (A3.39).

APPENDIX 4

In this appendix it will be proved that when $a \leqq r \leqq b$ and $t > 0$, the integrals

$$I_1 = \int \frac{\omega_s \bar{X}_1^{rr}(r,s) \, e^{st} \, ds}{s(s^2 + \omega_s^2)\left[R_r + \mu_o c \, \bar{X}_1^{rr}(s)\right]} \qquad \text{... A4.1}$$

$$I_2 = \int \frac{\bar{X}_1^{rr}(r,s) e^{st} \, ds}{(s^2 + \omega_s^2)\left[R_r + \mu_o c \bar{X}_1^{rr}(s)\right]} \qquad \text{... A4.2}$$

vanish in the limit over the arc FGE of Fig. (9.1).

Proof: For s tending to infinity and ω_s small

$$s(s^2 + \omega_s^2) \doteq s^3$$

and

$$s^2 + \omega_s^2 \doteq s^2$$

Furthermore, since $\dot{R}_r = 0$ and $\mu_o c \gg 0$ for a path where $\left|\bar{X}_1^{ss}(s)\right| > 0$

the function

$$R_r + \mu_o c \, \bar{X}_1^{rr}(s) \doteq \mu_o c \, \bar{X}_1^{rr}(s)$$

The problem is thus reduced to show that for $a \leqq r \leqq b$ and $t > 0$,

the integral

$$I = \int \frac{1}{\mu_o c s^m} \frac{\bar{X}_1^{rr}(r,s)}{\bar{X}_1^{rr}(s)} e^{st} \, ds \qquad \text{... A4.3}$$

$$m = 2 \text{ or } 3$$

vanishes in the limit over the required arc such that on this path

$$\left| \overline{X}_1^{ss}(s) \right| > 0.$$

Using the asymptotic expansions of the Bessel functions and following the procedure as in appendix 3, it may be shown that for s tending to infinity,

$$\overline{X}_1^{rr}(r,s) = \frac{\left[K'_1(sb/c) \, I_1(sr/c) - I'_1(sb/c) \, K_1(sr/c) \right]}{\left[K'_1(sa/c) \, I'_1(sb/c) - I'_1(sa/c) K'_1(sa/c) \right]}$$

$$= \sqrt{\frac{a}{r}} \; \frac{\cosh \; s(b-r)/c}{\sinh \; s(b-a)/c} \qquad \qquad \ldots \; A4.4$$

and

$$\overline{X}_1^{rr}(s) = \frac{\left[K'_1(sb/c) \, I_1(sa/c) - I'_1(sb/c) \, K_1(sa/c) \right]}{\left[K'_1(sa/c) \, I'_1(sb/c) - I'_1(sa/c) \, K'_1(sb/c) \right]}$$

$$= \frac{\left[\cosh \; s(b-a)/c \right]}{\left[\sinh \; s(b-a)/c \right]} \qquad \qquad \ldots \; A4.5$$

From which

$$\frac{\overline{X}_1^{rr}(r,s)}{\overline{X}_1^{rr}(s)} = \sqrt{\frac{a}{r}} \; \frac{\cosh \; [s(b-r)/c]}{\cosh \; [s(b-a)/c]} \qquad \qquad \ldots \; A4.6$$

Choose the radius of the circular arc $R = \frac{\Pi vc}{b-a}$ with points on the circumference given by $s = Re^{j\phi}, \; -\Pi \le \phi \le \Pi$. From equations (A4.5) and (A4.6) only a few steps will be needed to prove that over the contour defined by R, the function $\left| X_1^{rr}(s) \right| > o$ and the factor

$$\left| \frac{\overline{X}_1^{rr}(r,s)}{\overline{X}_1^{rr}(s)} \right| = \sqrt{\frac{a}{r}} \; \left| \frac{\cosh \; [s(b-r)/c]}{\cosh \; [s(b-a)/c]} \right| < c$$

showing that it is bounded. Furthermore, the location of the νth zero of the function $R_r + \mu_o c \bar{x}_1^{rr}(s) = 0$, defined by equation (A4.34) is at a distance of

$$R_\nu \doteq \sqrt{\left[\frac{R_r}{\mu_o(b-a)}\right]^2 + \left[\frac{\pi c\,(2\nu-1)}{2(b-a)}\right]^2} < \frac{\pi \nu c}{b-a}$$

from the origin. Hence, the arc does pass through this zero and all other zeroes having smaller magnitudes (including that at $s = -\frac{R_r}{L_r}$). Therefore, the absolute value of I

$$|I| = \left| \int_{\mu_o} \frac{X_1^{rr}(r,s)e^{st}}{cs^m \bar{X}_1^{rr}(s)}\,ds \right|$$

$$< R_m R^{-m+1} \int e^{tR\cos\phi}\,d\phi$$

where, the relations $\left|\frac{1}{s^m}\right| = R^{-m}$, $|ds| = Rd\phi$ and $\left|e^{st}\right| = e^{tR\cos\phi}$

have been used and R_m is a constant which is bounded.

Now for the path FF' and EF'

$$tR\cos\phi \le \gamma t, \gamma > o$$

so that

$$\left|I_{FF'}\right| = \left|I_{EE'}\right| < R_m R^{-m+1}e^{\gamma t} \int_{\phi_1}^{\pi/2} d\phi$$

$$= R_m R^{-m+1}e^{\gamma t}\,\sin^{-1}\frac{\gamma}{R}$$

$$= R_m\left[\frac{b-a}{\pi \nu c}\right]^{m-1}e^{\gamma t}\sin^{-1}\frac{\gamma(b-a)}{\pi \nu c}$$

$$\ldots \text{A4.7}$$

Hence

$$\underset{\nu \to \infty}{\text{Lim}} \left| I_{FF'} \right| = \underset{\nu \to \infty}{\text{Lim}} \left| I_{EE'} \right| = 0$$

... A4.8

For the path F'G and E'G

$$\left| I_{F'G} \right| = \left| I_{E'G} \right| < R_m R^{-m+1} \int_{\pi/2}^{\pi} e^{tR\cos\phi} \, d\phi$$

$$= R_m R^{-m+1} \int_{0}^{\pi/2} e^{-tR\sin\phi} \, d\phi$$

$$< R_m R^{-m+1} \int_{0}^{\pi/2} e^{-2tR\phi/\pi} \, d\phi$$

$$< R_m R^{-m} \frac{\pi}{2t} \, , \quad t > 0$$

$$= R_m \left[\frac{b-a}{\pi\nu c} \right]^m \frac{\pi}{2t}, \quad t > 0$$

... A4.9

This shows that

$$\underset{\nu \to \infty}{\text{Lim}} \left| I_{F'G} \right| = \underset{\nu \to \infty}{\text{Lim}} \left| I_{E'G} \right| = 0$$

... A4.10

Therefore, as $\nu \to \infty$, the integrals of equations (A4.1) and (A4.2) vanish over the arc FGE of Fig. (9.1).

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